Tema 3. Convoluciones continuas y discretas

Ejemplos de cálculo gráfico

Ingeniería de Telecomunicación

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2. Convoluciones continuas
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Convolución discreta. Definición y propiedades

Definición

$$y[n] = x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$$

Propiedades

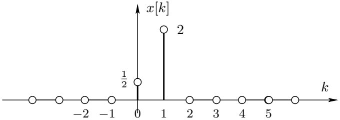
- Elemento neutro: $x[n] * \delta[n] = x[n]$
- Conmutativa: $x[n] * h[n] = h[n] * x[n]$
- Asociativa:
  $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n] = x[n] * h_1[n] * h_2[n]$
- Distributiva: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

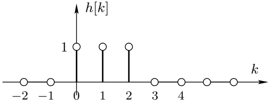
Ejemplo 1

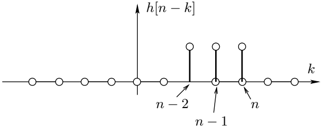
$x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$

$h[n] = u[n] - u[n-3]$

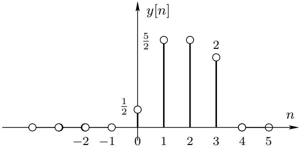
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$







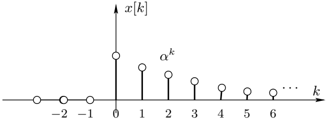
- $n < 0,\ y[n] = 0$
- $n = 0,\ y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = x[0]h[0] = \frac{1}{2}$
- $n = 1,\ y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = \frac{5}{2}$
- $n = 2,\ y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = x[0]h[2] + x[1]h[1] = \frac{5}{2}$
- $n = 3,\ y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = x[1]h[2] = 2$
- $n > 3,\ y[n] = 0$

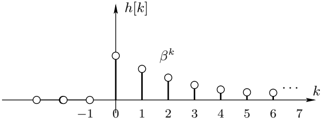


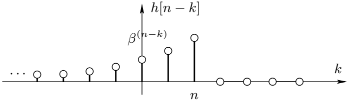
Ejemplo 2

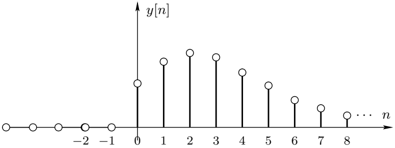
$$x[n] = \alpha^n u[n], \qquad \alpha \neq \beta,$$
$$h[n] = \beta^n u[n], \qquad 0 < \alpha, \beta < 1.$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{n-k} u[n-k] = \sum_{k=0}^{\infty} \alpha^k \beta^{n-k} u[n-k].$$









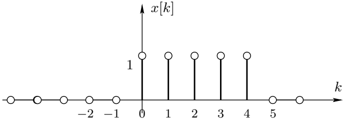
- $n < 0, \; y[n] = 0$
- $n \geq 0, \; y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}$

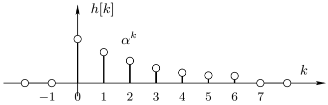
$$\boxed{y[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]}$$

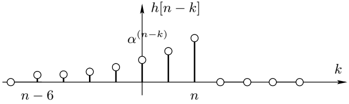
Ejemplo 3

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{resto} \end{cases}, \quad h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{resto} \end{cases}$$

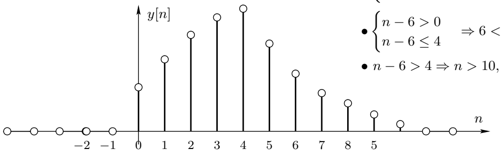
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$







- $n < 0,\ y[n] = 0$
- $0 \le n \le 4,\ y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$
- $\begin{cases} n > 4 \\ n - 6 \le 0 \end{cases} \Rightarrow 4 < n \le 6,\ y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$
- $\begin{cases} n - 6 > 0 \\ n - 6 \le 4 \end{cases} \Rightarrow 6 < n \le 10,\ y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$
- $n - 6 > 4 \Rightarrow n > 10,\ y[n] = 0$

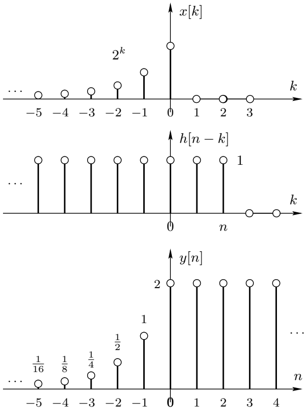


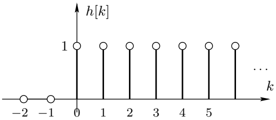
Ejemplo 4

$x[n] = 2^n u[-n]$

$h[n] = u[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

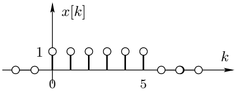


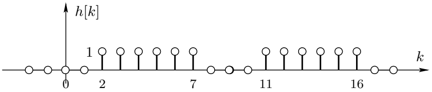


- $n < 0,\ y[n] = \sum_{k=-\infty}^{n} 2^k = 2^{n+1}$
- $n \geq 0,\ y[n] = \sum_{k=-\infty}^{0} 2^k = 2$

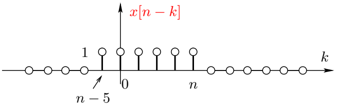
Ejemplo 5

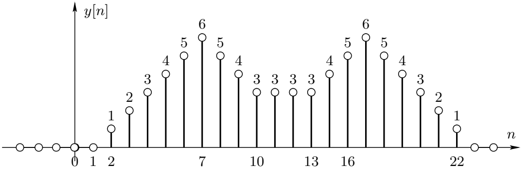
$$x[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{resto} \end{cases}, \quad h[n] = \begin{cases} 1, & 2 \leq n \leq 7, 11 \leq n \leq 16 \\ 0, & \text{resto} \end{cases}$$



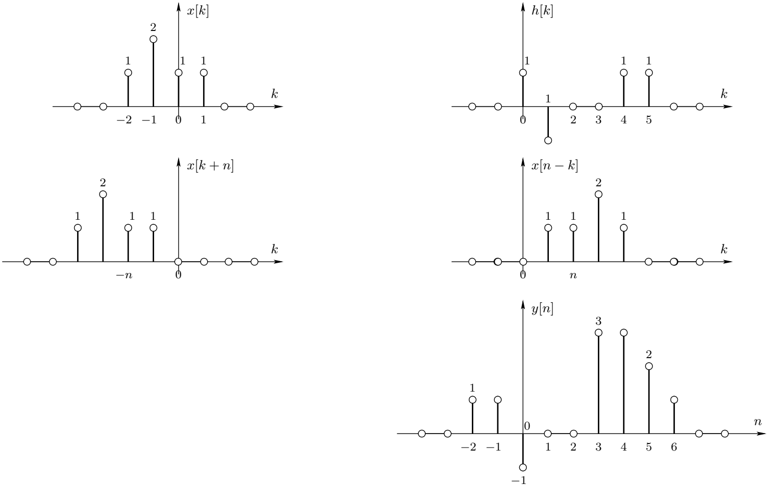


- $n < 2, \; y[n] = 0$
- $2 \leq n \leq 7, \; y[n] = \sum_{k=2}^{n} 1 = n - 1$
- $8 \leq n \leq 10, \; y[n] = \sum_{k=n-5}^{7} 1 = 13 - n$
- $11 \leq n \leq 12, \; y[n] = 3$
- $13 \leq n \leq 16, \; y[n] = \sum_{k=11}^{n} 1 = n - 10$
- $17 \leq n \leq 21, \; y[n] = \sum_{k=n-5}^{16} 1 = 22 - n$
- $n \geq 22, \; y[n] = 0$





Ejemplo 6



x[k]
2
1
1
1
k
−2
−1
0
1
h[k]
1
1
1
1
k
0
2
3
4
5
x[k + n]
2
1
1
1
k
−n
0
x[n − k]
2
1
1
1
k
0
n
y[n]
3
2
1
0
n
−2
−1
1
2
3
4
5
6
−1

Definición

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Propiedades

- Elemento neutro: $x(t) * \delta(t) = x(t)$
- Conmutativa: $x(t) * h(t) = h(t) * x(t)$
- Asociativa:

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t) = [x(t) * h_2(t)] * h_1(t)$$

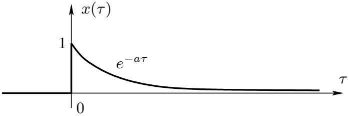
- Distributiva: $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

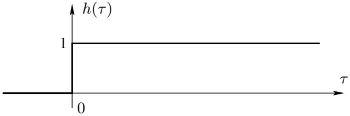
Convolución continua. Ejemplos

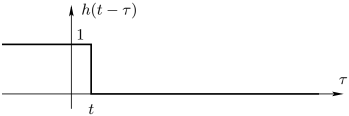
Ejemplo 1

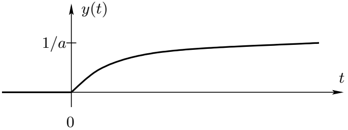
$x(t) = e^{-at}u(t),\ a > 0$
$h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$







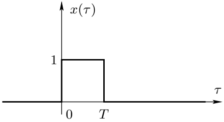


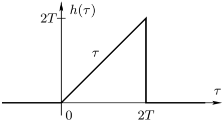
- $t < 0,\ y(t) = 0$
- $t \geq 0,\ y(t) = \int_0^t e^{-a\tau}d\tau = \frac{1-e^{-at}}{a}$

$$\boxed{y(t) = \frac{1-e^{-at}}{a}u(t),\ \forall t}$$

Ejemplo 2

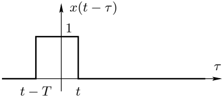
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{resto} \end{cases}, \qquad h(t) = \begin{cases} t, & 0 < t < T \\ 0, & \text{resto} \end{cases}$$

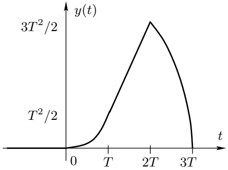




- $t \leq 0, \; y(t) = 0$
- $\begin{cases} t > 0 \\ t - T \leq 0 \end{cases} \Rightarrow 0 < t \leq T, \; y(t) = \int_0^t \tau d\tau = \frac{t^2}{2}$
- $\begin{cases} t - T > 0, \\ t \leq 2T \end{cases} \Rightarrow T < t \leq 2T, \; y(t) = \int_{t-T}^t \tau d\tau = tT - \frac{1}{2}T^2$
- $\begin{cases} t > 2T, \\ t - T \leq 2T \end{cases} \Rightarrow 2T < t \leq 3T, \; y(t) = \int_{t-T}^{2T} \tau d\tau = tT - \frac{1}{2}t^2 + \frac{3}{2}T^2$
- $t - T > 3T \rightarrow t > 3T, \; y(t) = 0$

$$y(t) = \frac{t^2}{2}u(t) - \frac{1}{2}(t-T)^2 u(t-T) + \left(2T^2 - \frac{1}{2}t^2\right) u(t-2T) + \\ + \left(\frac{1}{2}t^2 - tT - \frac{3}{2}T^2\right) u(t-3T)$$

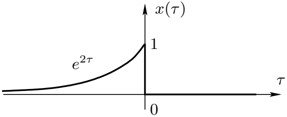


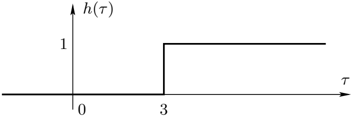


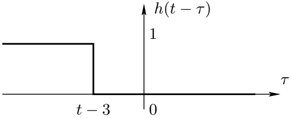
Ejemplo 3

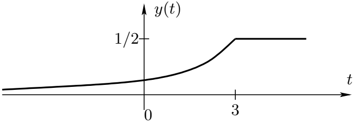
$x(t) = e^{2t}u(-t)$
$h(t) = u(t-3)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$









- $t-3 < 0 \Rightarrow t < 3,\ y(t) = \int_{-\infty}^{t-3} e^{2\tau}d\tau = \frac{1}{2}e^{2(t-3)}$
- $t-3 \geq 0 \Rightarrow t \geq 3,\ y(t) = \int_{-\infty}^{0} e^{2\tau}d\tau = \frac{1}{2}$

$$y(t) = \frac{1}{2}e^{2(t-3)} + \frac{1}{2}\left[1 - e^{2(t-3)}\right]u(t-3)$$