Noise estimation and diffusion signal reconstruction: From cradle to parallel imaging

What type of noise ‘infects’ the data and by filtering it out are we (black) magically creating something new?

Santiago Aja-Fernández

Laboratory of Image Processing

Universidad de Valladolid

Illustrations: David Aja
Thinking about the problem
Noise in MR data: An aesthetic problem?

Diffusion tensor field over FA

- Noise is one of the main sources of quality deterioration in magnetic resonance (MR) data.
- Is noise just a problem for “image quality" and visual inspection?
- Affecting: segmentation, registration, tensor estimation...
- In dMRI: noise and filtering may affect the estimation of direction and amount of diffusion.
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MR filtering

We can clean the images... is it enough in dMRI?

Laboratory of Image Processing (LPI)

Noise and signal estimation
MR filtering

We can clean the images... is it enough in dMRI?
Diffusion Tensor, real example

Diffusion tensor field over FA

Without filtering

LMMSE filtered

Q-Balls imaging, real example
Q-Balls imaging, real example

Comparison: Q-Balls, DOT, OPDT

Without LMMSE-\(N\) filtering
Q-Balls imaging, real example

Comparison: Q-Balls, DOT, OPDT

With LMMSE-N filtering
Noise is known to be one of the main sources of quality deterioration in magnetic resonance (MR) data.

We want to get rid of that noise but preserving the underlying structures (very important in dMRI).

Accordingly:
- Filtering methods based on data structure and modeling of noise behavior. Bayesian and probabilistic modeling.
- Quality assessment methods to test the goodness of proposed algorithms.
- Estimation of parameters out of data: variance of noise estimation.
- Filtering and preprocessing: model based.
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1  Introduction
2  Signal and Noise statistical models
3  Noise filtering and signal estimation
4  Noise estimation
5  Effects on dMRI
6  Pitfalls and conclusions
1 Introduction

2 Signal and Noise statistical models

3 Noise filtering and signal estimation

4 Noise estimation

5 Effects on dMRI

6 Pitfalls and conclusions
Formation of MR images (simple model)

k-space
- Complex Gaussian noise
- Uncorrelated
- Stationary

x-space
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Magnitude
- Magnitude of complex Gaussian: Rician.
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scanner → k-space → $F^{-1}$ → x-space → $\cdot$ → magnitude
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scanner → $k$-space $\xrightarrow{F^{-1}}$ $x$-space $\rightarrow$ magnitude

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Areas in the image

Rician Area

Rayleigh Area
Signal and noise statistical models in MR

### Before Magnitude

<table>
<thead>
<tr>
<th>k-space</th>
<th>x-space</th>
<th>Complex Gaussian Complex Gaussian</th>
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### Composite Magnitude Image

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<th>Acquisition</th>
<th>Statistical Model</th>
<th>Stat. model of the background</th>
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<td>Rician (Stationary)</td>
<td>Rayleigh</td>
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<tr>
<td>Multiple coils</td>
<td>No subsampling+ SoS</td>
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For high SNR: always possible to use **Gaussian** assumption.
Stationarity (brief, quick and intuitive)

Variance of noise

- Stationary: same $\sigma_n^2$ value for every pixel.
- Non-Stationary: $\sigma_n^2$ varies along the image.
Too many abstract concepts...

Let's go back to earth!
Too many abstract concepts...

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Outline

1. Introduction
2. Signal and Noise statistical models
3. Noise filtering and signal estimation
4. Noise estimation
5. Effects on dMRI
6. Pitfalls and conclusions
Purpose: eliminate the noise in MR data without destroying any signal information.

Basically: we want to improve the SNR of our data.

In dMRI special attention to noise models: filtering may introduce bias.

Trade off between denoising and structure keeping.

REMEMBER: We are not *inventing* data or *cleaning* an image; we are estimating a signal out of noisy data. Ideally: we are recovering the most likely or possible signal based on the data we have.
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**Alternative:** filtering in the complex domain (scanner) using Gaussian model.
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Example: LMMSE Signal estimation

Signal estimation: the LMMSE estimator

LMSSE estimator:

\[ \hat{\theta} = E\{\theta\} + C_{\theta x}C_{xx}^{-1}(x - E\{x\}) \]

Rewriting for a 2D signal with a Rician distribution

\[ \hat{A}_{ij}^2 = E\{A_{ij}^2\} + C_{A_{ij}^2 M_{ij}^2}C_{M_{ij}^2 M_{ij}^2}^{-1}(M_{ij}^2 - E\{M_{ij}^2\}) \]

From here the estimator becomes:

\[ \hat{A}^2(x) = \langle M(x)^2 \rangle - 2\sigma_n^2 + K(x) \left( M^2(x) - \langle M(x)^2 \rangle \right), \]

with

\[ K(x) = 1 - \frac{4\sigma_n^2 (\langle M(x)^2 \rangle - \sigma_n^2)}{\langle M(x)^4 \rangle - \langle M(x)^2 \rangle^2}. \]
Examples: quality measures

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<td>124.44</td>
<td>0.7770</td>
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<td>0.9365</td>
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<td>0.8792</td>
<td>0.9210</td>
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<td>153.81</td>
<td>0.8412</td>
<td>0.6483</td>
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<td>0.7197</td>
<td>80.21</td>
<td>0.8924</td>
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<tr>
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<td>72.40</td>
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<td>0.9118</td>
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<tr>
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<td>46.83</td>
<td>0.9242</td>
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<tr>
<td>ORNRAD</td>
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<td>0.9824</td>
<td><strong>26.96</strong></td>
<td><strong>0.9432</strong></td>
<td>0.9692</td>
</tr>
</tbody>
</table>
Examples: color by orientation

Original  UNLM-5  LMMSE-1  LMMSE-15
Outline

1. Introduction
2. Signal and Noise statistical models
3. Noise filtering and signal estimation
4. Noise estimation
5. Effects on dMRI
6. Pitfalls and conclusions
Many filtering methods require an estimation of $\sigma_n^2$ (variance of noise).

Variance of noise can be measure of quality in the data.

Not only for filtering: Tensor estimation, segmentation methods based on the Rician distribution and fiber orientation estimators.
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Not only for filtering: Tensor estimation, segmentation methods based on the Rician distribution and fiber orientation estimators.
What do we want to estimate?

Rician distribution:

\[ p_M(M|A, \sigma_n) = \frac{M}{\sigma_n^2} e^{-\frac{M^2+A^2}{2\sigma_n^2}} I_0 \left( \frac{AM}{\sigma_n^2} \right) u(M), \]

Rayleigh distribution

\[ p_M(M|\sigma_n) = \frac{M}{\sigma_n^2} e^{-\frac{M^2}{2\sigma_n^2}} u(M). \]

We want to estimate \( \sigma_n^2 \), the variance of noise in the complex \( x \)-space:

\[ C(x) = A(x) + N(x; \sigma_n^2) \]

with

\[ N(x, \sigma_n^2) = N_r(x; \sigma_n^2) + j \cdot N_i(x; \sigma_n^2) \]
How to estimate?

- We define an estimator: usually related to statistics of the image.
- E.g.: Mean of the (Rayleigh) background

\[ E\{r\} = \sigma \sqrt{\frac{\pi}{2}} \]

So we can define

\[ \sigma = \sqrt{\frac{2}{\pi}} E\{r\} \quad \rightarrow \quad \hat{\sigma}_n = \sqrt{\frac{2}{\pi}} \langle M(x_B) \rangle \]

- Disadvantages: need of background segmentation; assumption of uniform background; sensitive to errors and artifacts.
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How to estimate? An example

(a) Second order moment

(b) Sample mean

- Using the mode of some distribution (most probable value).
- Distribution of sample second order moment of Rayleigh data: Gamma distribution

\[ S = \frac{1}{N} \sum_{i=1}^{N} R_i^2(\sigma^2) \sim \gamma\left(N, \frac{2\sigma^2}{N}\right) \]

with maximum in \((N-1)/N \cdot 2\sigma^2_n\).

So:

\[ \hat{\sigma}^2_n = \frac{N}{N-1} \cdot \frac{1}{2} \text{mode}\{ \langle M^2(x) \rangle \} \approx \frac{1}{2} \text{mode}\{ \langle M^2(x) \rangle \} \]

- Advantages: robust, not need of segmentation
- Disadvantages: problems to calculate the mode, artificial backgrounds.
How to estimate? An example

(a) Second order moment

![Graph showing second order moment for different window sizes (3 x 3, 7 x 7, 11 x 11, 21 x 21).]

(b) Sample mean

![Graph showing sample mean for different window sizes (3 x 3, 7 x 7, 11 x 11, 21 x 21).]

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Methods estimating from global statistic of segmented background or selected area of the background. (Rayleigh assumption)

Methods using the mode of the distribution of a local statistic. Robust and fast, no segmentation needed. (Rayleigh assumption)

Methods matching a known distribution to a sample distribution (EM, ML, maximization or certain parameter...). Robust, but recursive estimation.

Methods based on wavelets: most of them are implicitly assuming a Gaussian distribution.

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The error (MSE) for multiple-coil is defined as

\[
\text{MSE} \approx \left[ \frac{K_1}{N} \left( \frac{1}{\text{SNR}^2} - \frac{1}{\text{SNR}^4} (3L - 4) \right) \right] + \left[ \frac{1}{\text{SNR}^4} 3(L - 1)^2 \right]
\]
Synthetic experiments

(a) Original

(b) Rician

(c) pMRI-SoS

(d) SENSE

(e) GRAPPA
Estimation tensor: Synthetic experiments

(a) Rician  (b) pMRI-SoS  (c) SENSE  (d) GRAPPA
Estimation tensor: Synthetic experiments

(a) Noisy

(b) Gaussian

(c) Wiener

(d) LMMSE

(e) RLMMSE (5)
A realistic DWI phantom is used, [Tristan09b]. A $256 \times 256 \times 81$ volume, spatial resolution of $1\text{mm} \times 1\text{mm} \times 1.7\text{mm}$, 15 gradient directions and 1 baseline.

rMSE of the tensor estimation for (Left) different $\sigma_n$ values (and 5 gradients); (Right) different number of gradients (and $\sigma_n = 10$)

$$
\text{rMSE}(x) = \frac{\sqrt{(\hat{\lambda}_1(x) - \lambda_1(x))^2 + (\hat{\lambda}_2(x) - \lambda_2(x))^2 + (\hat{\lambda}_3(x) - \lambda_3(x))^2}}{\lambda_1(x)}
$$
Realistic DWI phantom

Fractional Anisotropy. From left to right: Original non-noisy data; Rician case; pMRI-SoS case; SENSE case; GRAPPA case. Top row $\sigma_n = 10$ (average SNR in gray matter in the gradient images 40). Low row: $\sigma_n = 35$ (average SNR in gray matter in the gradient images 11.4).
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Pitfalls and conclusions

Pitfalls

- Acquisition: reduced $k$-space and EPI introduces non-linearity that make the signal differs from model.
- Correlations must be taking into account.
- Parallel acquisition: Non-stationary model. Is noise estimation possible? Has it any meaning?

Conclusions

- Noise affects not only the visual quality but the estimation of diffusion parameters.
- Knowing the underlying noise model helps to better filtering.
- Proper noise estimation improves signal estimation (and noise filtering).
- Better to filter BEFORE estimation.


Thanks for you attention!
Noise estimation and diffusion signal reconstruction: From cradle to parallel imaging

What type of noise ‘infects’ the data and by filtering it out are we (black) magically creating something new?

Santiago Aja-Fernández

Laboratory of Image Processing

Universidad de Valladolid

Illustrations: David Aja