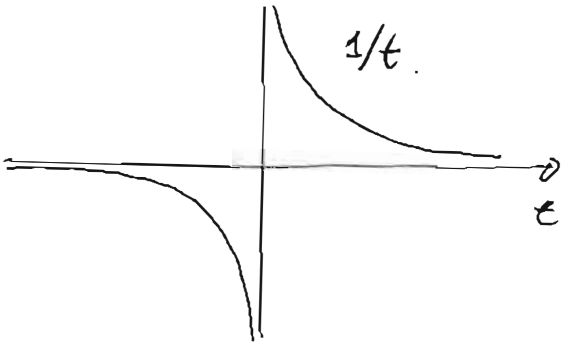
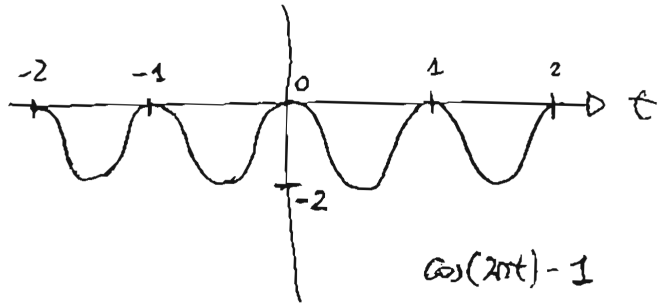
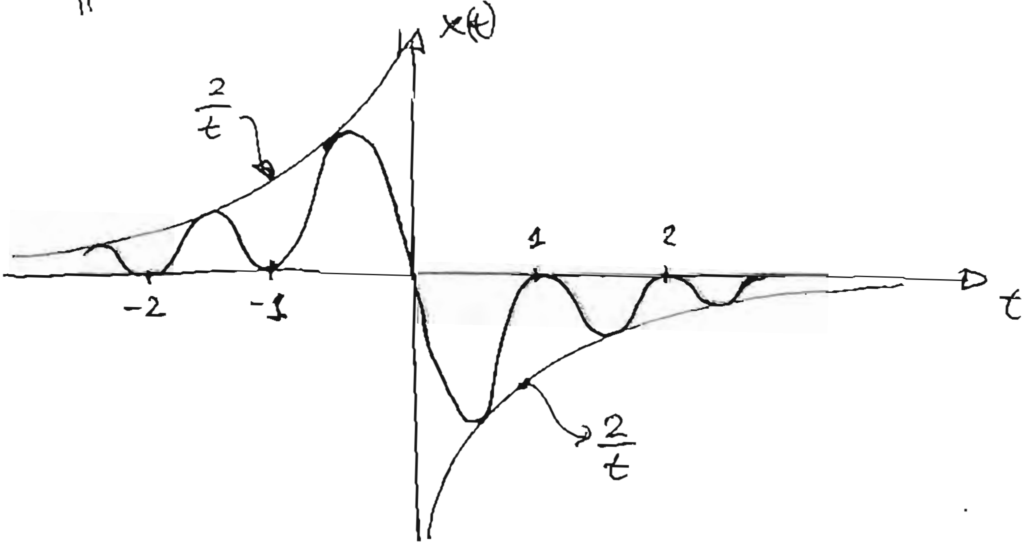
## PRIMER PROBLEMA

a) $x(t) = \dfrac{\cos(2\pi t) - 1}{t} = \dfrac{1}{t}\left(\cos(2\pi t) - 1\right)$

para $t = 0$ $\quad x(0) = \frac{0}{0}$ por L'Hopital $\quad \dfrac{-\operatorname{sen}(2\pi t) \cdot 2\pi}{1} = 0 \Rightarrow x(0) = 0.$



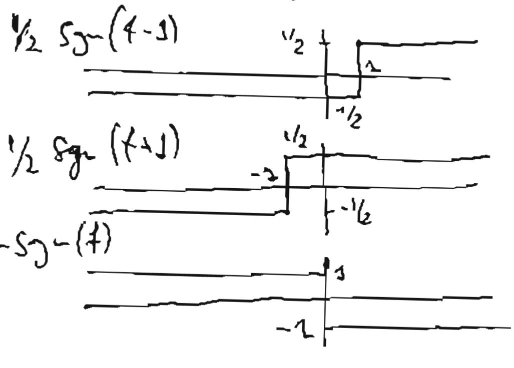

b) $x(t) = \left[\cos(2\pi t) - 1\right]\dfrac{\pi}{\pi t} = \pi\left[\cos(2\pi t) - 1\right]\dfrac{1}{\pi t} = \pi\left[\cos(2\pi t)\right]\dfrac{1}{\pi t} - \pi\dfrac{1}{\pi t}$

Sabemos por TF inmediatas:

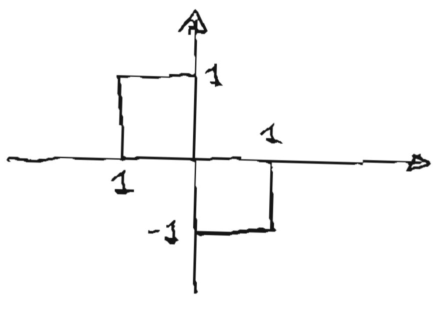
$$1 \leftrightarrow \delta(f) \qquad \cos(2\pi t) \leftrightarrow \frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1) \quad y \quad \frac{1}{\pi t} \leftrightarrow -j\,\mathrm{sgn}(f)$$

$$X(f) = \pi\left[\frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)\right] * \left[-j\,\mathrm{sgn}(f)\right] - \pi\left[-j\,\mathrm{sgn}(f)\right]$$

$$= -j\pi\left[\frac{1}{2}\mathrm{sgn}(f-1) + \frac{1}{2}\mathrm{sgn}(f+1) - \mathrm{sgn}(f)\right]$$

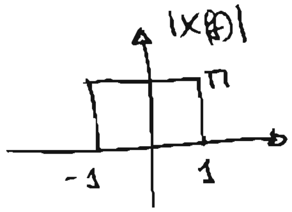


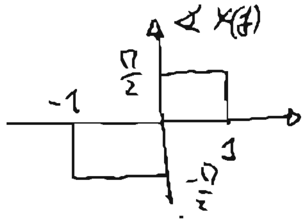

sumando $\Rightarrow$



$$X(f) = j\pi\,\Pi\left(\frac{f}{2}\right)\text{sgn}(f) = j\pi\left[\Pi(f-1/2) - \Pi(f+1/2)\right]$$

$$|X(f)| = \pi \cdot \Pi(f/2) \quad \text{y} \quad \measuredangle X(f) = \frac{\pi}{2}\Pi\left(\frac{f}{2}\right)\text{sgn}(f) = \frac{\pi}{2}\left[\Pi(f-1/2) - \Pi(f+1/2)\right]$$




c) $\hat{x}(t)$ por TH inmediatas: $\frac{\text{sen}(t)}{t} \longleftrightarrow \frac{1-\cos(t)}{t}$.

propiedad TH: $TH\{\hat{x}(t)\} = -x(t)$, entonces,

$$\frac{1-\cos(t)}{t} \longleftrightarrow -\frac{\text{sen}(t)}{t}$$

Como TH es lineal, multiplicando por constante $-1$:

$$x_1(t) = \frac{\cos(t)-1}{t} \longleftrightarrow \hat{x}_1(t) = \frac{\text{sen}(t)}{t}.$$

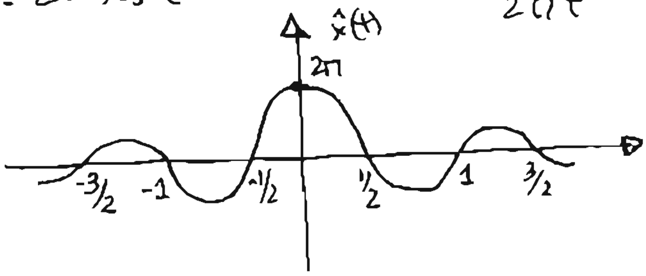
pero yo quiero:

$$x(t) = \frac{\cos(2\pi t)-1}{t} = 2\pi\frac{\cos(2\pi t)-1}{2\pi t} = 2\pi\, x_1(2\pi t) \quad \text{escalado en tiempo}$$

$$\hat{x}(t) = \int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\,d\tau = \int_{-\infty}^{\infty}\frac{2\pi\, x_1(2\pi\tau)}{t-\tau}\,d\tau \quad \Bigg| \quad \text{Cambio variable } \tau' = 2\pi\tau, \; d\tau' = 2\pi d\tau$$

$$= \int_{-\infty}^{\infty}\frac{x_1(\tau')}{t-\frac{\tau'}{2\pi}}\,d\tau' = 2\pi\int_{-\infty}^{\infty}\frac{x_1(\tau')}{2\pi t-\tau'}\,d\tau' \quad \Bigg| \quad \text{como } \hat{x}_1(t) = \int_{-\infty}^{\infty}\frac{x_1(\tau')}{t-\tau'}\,d\tau'$$

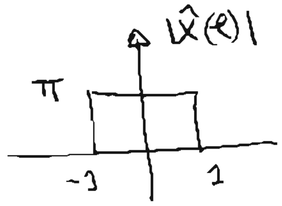
entonces $\boxed{\hat{x}(t) = 2\pi\,\hat{x}_1(2\pi t) = 2\pi\frac{\text{sen}(2\pi t)}{2\pi t} = \frac{\text{sen}(2\pi t)}{t} = 2\pi\,\text{sinc}(2t)}$

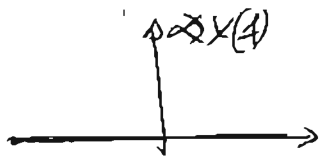


d) $\hat{X}(f)$ sabemos por TF inmediatas:

$$2\,\mathrm{sinc}(2t) \longleftrightarrow \Pi\left(\frac{f}{2}\right)$$

$$\hat{X}(f) = \pi \cdot \Pi\left(\frac{f}{2}\right) \Rightarrow |\hat{X}(f)| = \pi \cdot \Pi\left(\frac{f}{2}\right) \quad \measuredangle X(f) = 0.$$



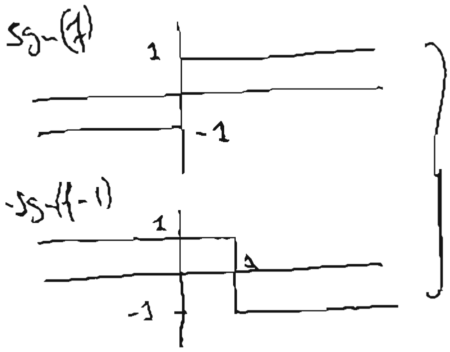




e) $\boxed{x_+(t)} = x(t) + j\hat{x}(t) = \frac{\cos(2\pi t) - 1}{t} + j\,\frac{\mathrm{sen}(2\pi t)}{t} = \boxed{\frac{e^{j2\pi t} - 1}{t}}$

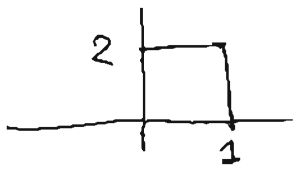
f) $X_+(f) \Rightarrow \quad x_+(t) = \pi\left[e^{j2\pi t} - 1\right]\frac{1}{\pi t}$

$$X_+(f) = \pi\left[\delta(f-1) - \delta(f)\right] * \left[-j\,\mathrm{sgn}(f)\right] =$$

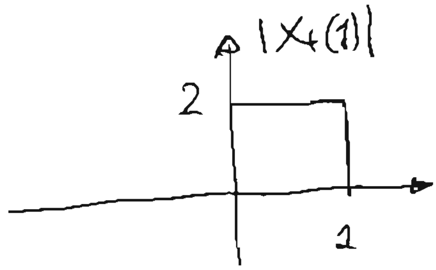
$$= +j\pi\left[\delta(f) - \delta(f-1)\right] * \mathrm{sgn}(f) = j\pi\left[\mathrm{sgn}(f) - \mathrm{sgn}(f-1)\right]$$

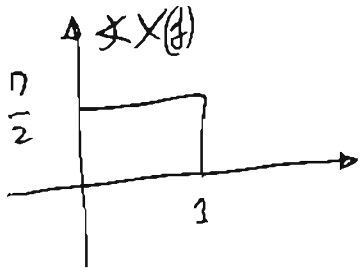



sumando $\Rightarrow$




$$X_+(f) = j2\pi\,\Pi\left(f - \frac{1}{2}\right) \Rightarrow \begin{cases} |X_+(f)| = 2\pi\,\Pi\left(f - \frac{1}{2}\right) \\ \measuredangle X_+(f) = \frac{\pi}{2}\,\Pi\left(f - \frac{1}{2}\right) \end{cases}$$



g) $\tilde{x}(t)$ por apartado anterior, la frecuencia a mitad de banda $\rightarrow f_c = 1/2$.

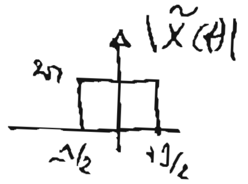
$$\tilde{X}(t) = x_+(t)e^{-j2\pi f_c t} = \frac{e^{j\pi t}-1}{t}\, e^{-j\pi t} = \frac{e^{j\pi t}-e^{-j\pi t}}{t} = 2j\,\frac{\operatorname{sen}(\pi t)}{t}$$

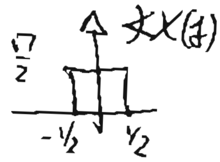
$$\boxed{\tilde{x}(t) = 2\pi j\,\operatorname{sinc}(t)}$$

h) $\tilde{X}(f) = 2\pi j\,\Pi(f)$

$|\tilde{X}(f)| = 2\pi\,\Pi(f)$

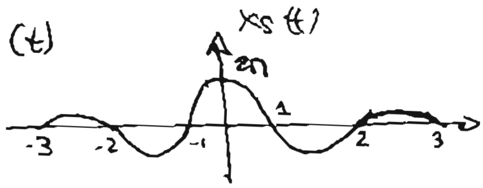
$\measuredangle \tilde{X}(f) = \frac{\pi}{2}\,\Pi(f)$



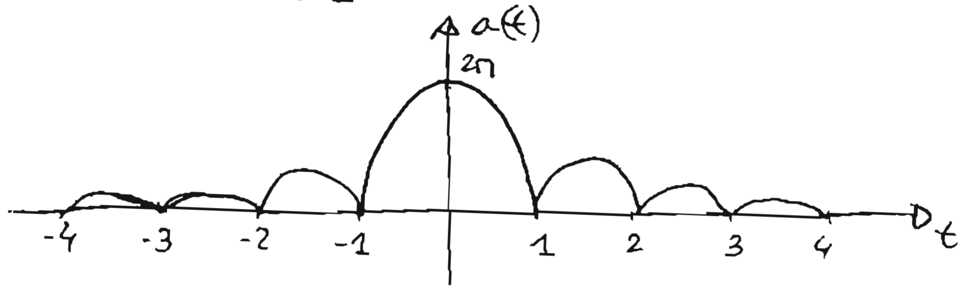

i) $\tilde{x}(t) = x_c(t) + j x_s(t) \Rightarrow$ $x_c(t) = 0$

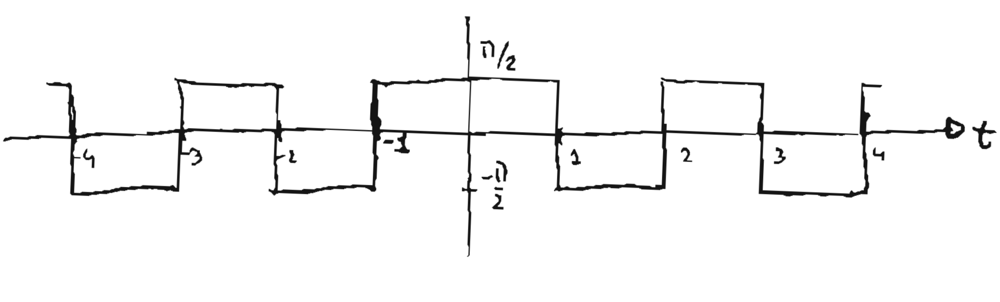
$x_s(t) = 2\pi\,\operatorname{sinc}(t)$


j) $a(t) = \sqrt{x_c^2(t) + x_s^2(t)} = 2\pi\sqrt{\operatorname{sinc}^2(t)} = 2\pi\,|\operatorname{sinc}(t)|$

$$\phi(t) = \operatorname{atan}\frac{x_s(t)}{x_c(t)} = \begin{cases} \frac{\pi}{2} & \operatorname{sinc}(t) > 0 \\ -\frac{\pi}{2} & \operatorname{sinc}(t) < 0 \end{cases} = \frac{\pi}{2}\operatorname{sgn}[\operatorname{sinc}(t)]$$



SEGUNDO PROBLEMA

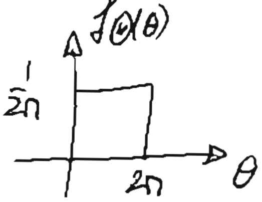
a) $m_X(t_n) = E[X(t_n)] = E[A(t_n)\, C(t_n)]$ por ser independientes

$$m_X(t_n) = E[A(t_n)] \cdot E[C(t_n)] = A_0\, E[C(t_n)]$$

$$E[C(t_n)] = E[C_0 \cdot \cos(2\pi f_0 t + \Theta)] = C_0\, E[\cos(2\pi f_0 t + \Theta)]$$

por ser $\Theta$ uniforme ente $(0, 2\pi)$

$$f_\Theta(\theta) = \frac{1}{2\pi} \Pi\left(\frac{\theta - \pi}{2\pi}\right)$$




$$E[\cos(2\pi f_0 t + \Theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_0 t + \theta)\, f_\Theta(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi f_0 t + \theta)\, d\theta =$$

$$\frac{1}{2\pi}\left[\operatorname{sen}(2\pi f_0 t + \theta\right]_0^{2\pi} = \frac{1}{2\pi}\left[\operatorname{sen}(2\pi f_0 t + 2\pi) - \operatorname{sen}(2\pi f_0 t)\right] =$$

$$= \frac{1}{2\pi}\left[\operatorname{sen}(2\pi f_0 t)\cos(2\pi) + \cos(2\pi f_0 t)\,\cancel{\operatorname{sen}(2\pi)} - \operatorname{sen}(2\pi f_0 t)\right] = 0$$

$$\boxed{m_X(t_n) = A_0 \cdot C_0 \cdot 0 = 0}$$

b) $R_X(t_n, t_m) = E[X(t_n) X(t_m)] = E[A(t_n) C(t_n) A(t_m) C(t_m)]$

por ser independientes

$$R_X(t_n, t_m) = E[A(t_n) A(t_m)]\, E[C(t_n) C(t_m)]$$

$$R_A(\tau) = R_A(t_m - t_n) = E[A(t_n + \tau) A(t_n)] = E[A(t_m) A(t_n)]$$ con $\tau = t_m - t_n$ $\forall t_n$

entonces

$$R_X(t_n, t_m) = R_A(t_m - t_n)\, E[C(t_n) C(t_m)] = R_A(t_m - t_n)\, R_C(t_n, t_m)$$

$$R_C(t_n,t_m) = E[C(t_n)C(t_m)] = E[C_0\cos(2\pi f_0 t_n+\Theta)\, C_0\cos(2\pi f_0 t_m+\Theta)] =$$ (2)

$$= C_0^2\, E[\cos(2\pi f_0 t_n+\Theta)\cos(2\pi f_0 t_m+\Theta)] = \frac{C_0^2}{2} E\Big[\cos[2\pi f_0(t_n+t_m)+2\Theta] + \cos[2\pi f_0(t_m-t_n)]\Big]$$

$$= \frac{C_0^2}{2}\cos[2\pi f_0(t_m-t_n)] + \frac{C_0^2}{2} E\Big[\cos[2\pi f_0(t_n+t_m)+2\Theta]\Big]$$

$$E[\cos[2\pi f_0(t_m+t_n)+2\Theta]] = \int_{-\infty}^{\infty}\cos[2\pi f_0(t_n+t_m)+2\theta]\, f_\Theta(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos[2\pi f_0(t_n+t_m)+2\theta]\,d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\operatorname{sen}[2\pi f_0(t_n+t_m)+2\theta]}{2}\right]_0^{2\pi} = \frac{\operatorname{sen}[2\pi f_0(t_n+t_m)+4\pi] - \operatorname{sen}[2\pi f_0(t_n+t_m)]}{4\pi} =$$

$$= \frac{\operatorname{sen}[2\pi f_0(t_n+t_m)]\cos(4\pi) + \cos[2\pi f_0(t_n+t_m)]\,\cancel{\operatorname{sen}(4\pi)} - \operatorname{sen}[2\pi f_0(t_n+t_m)]}{4\pi} = 0$$

$$R_C(t_n,t_m) = R_C(\tau) = \frac{C_0^2}{2}\cos[2\pi f_0(t_m-t_n)] = \frac{C_0^2}{2}\cos(2\pi f_0\tau), \quad \underline{\tau = t_m-t_n}$$

$$\boxed{R_X(t_n,t_m) = \frac{C_0^2}{2} R_A(t_m-t_n)\cos[2\pi f_0(t_m-t_n)]}$$

para $\tau = t_m - t_n$ $\qquad R_X(\tau) = \frac{C_0^2}{2} R_A(\tau)\cos(2\pi f_0\tau)$

c) MEDIA $\quad m_X = 0$ no depende de $t$.

AUTOCORRELACION $\quad R_X(\tau) = \frac{C_0^2}{2} R_A(\tau)\cos(2\pi f_0\tau)$ depende sólo de $\tau = t_m - t_n$.

$X(t)$ ES ESTACIONARIA EN SENTIDO AMPLIO.

$$S_X(f) = TF\{R_X(\tau)\} = TF\left\{\frac{C_0^2}{2} R_A(\tau)\cos(2\pi f_0\tau)\right\} =$$

$$\frac{C_0^2}{2} TF\{R_A(\tau)\} * \left\{\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right\}$$

definiendo la densidad espectral de potencia de $A(t)$ como

$$S_A(f) = TF\{R_A(\tau)\}$$

entonces

$$\boxed{S_X(f) = \frac{C_0^2}{4} S_A(f) * \{\delta(f-f_0) + \delta(f+f_0)\} = \frac{C_0^2}{4} S_A(f-f_0) + \frac{C_0^2}{4} S_A(f+f_0)}$$

d) Como $A(t)$ es Gaussiana $A_n = A(t_n)$ es una variable aleatoria Gaussiana. Para definir su función de densidad $f_{A_n}(a)$ sólo necesitamos saber su media y varianza.

$$\boxed{E[A_n] = E[A(t_n)] = A_0}.$$

Además el valor de la autocorrelación para $\tau = 0$:

$$R_A(0) = E[A^2(t_n)] = E[A_n^2] = \sigma_{A_n}^2 + A_0^2$$

$$\boxed{\sigma_{A_n}^2 = R_A(0) - A_0^2}$$

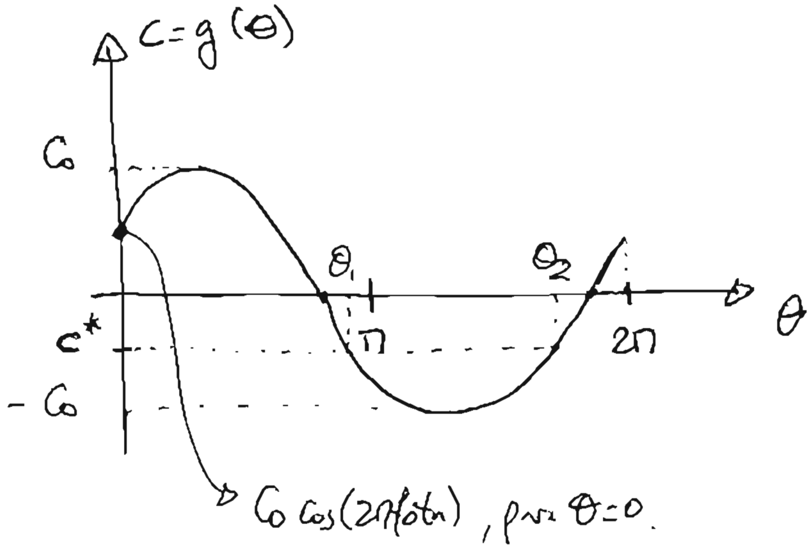
$$\boxed{f_{A_n}(a) = \frac{1}{\sqrt{2\pi(R_A(0) - A_0^2)}} \exp\left(-\frac{(a-A_0)^2}{2R_A(0) - 2A_0^2}\right)}$$

como sí es estacionario en sentido amplio (ya que tiene media constante $A_0$ y autocorrelación función de $\tau$) y además es Gaussiana, lo es en sentido estricto lo que se puede ver en que $f_{A_n}(a)$ no depende de $t$.

e) $C(t_n) = C_n$ transformación variable aleatoria:

$$\left.\begin{aligned} c &= g(\theta) = C_0 \cos(2\pi f_0 t_n + \theta) \\ C_n &= g(\Theta) = C_0 \cos(2\pi f_0 t_n + \Theta) \end{aligned}\right\} \quad \text{con} \quad f_\Theta(\theta) = \frac{1}{2\pi} \Pi\left(\frac{\theta - \pi}{2\pi}\right)$$

$c = g(\theta)$ función sólo de $\theta$, lo demás son constantes.




Cómo se puede ver en la gráfica, fijado un valor $c=c^*$, tenemos dos soluciones para $\theta=g^{-1}(c=c^*)$ que son $\theta_1$ y $\theta_2$.

El intervalo de definición de c es $(-C_0, C_0)$ (recorrido curva).

$$f_{C_n}(c) = \frac{f_\Theta(\theta_1 = g^{-1}(c=c^*))}{\left|\frac{dg(\theta)}{d\theta}\right|_{\theta_1=g^{-1}(c=c^*)}} + \frac{f_\Theta(\theta_2 = g^{-1}(c=c^*))}{\left|\frac{dg(\theta)}{d\theta}\right|_{\theta_2=g^{-1}(c=c^*)}}.$$

$f_\Theta(\theta)$ vale siempre $\frac{1}{2\pi}$ tanto para $\theta_1$ como para $\theta_2$.

Vamos a calcular la derivada:

$$\frac{dg(\theta)}{d\theta} = -C_0 \operatorname{sen}(2\pi f_0 t_n + \theta)$$ que hay que ponerlo como función de c y considerar los dos casos

$$\operatorname{sen}(2\pi f_0 t + \theta) = \pm\sqrt{1-\cos^2(2\pi f_0 t+\theta)} = \pm\sqrt{1-\left(\frac{c}{C_0}\right)^2} = \pm\frac{1}{C_0}\sqrt{C_0^2-c^2}$$

$$\frac{dg(\theta)}{d\theta} = \mp\sqrt{C_0^2-c^2}$$ pero al tomar módulo $\left|\frac{dg(\theta)}{d\theta}\right| = \sqrt{C_0^2-c^2}$

Entonces: $$f_{C_n}(c) = \frac{1/2\pi}{\sqrt{C_0^2-c^2}} + \frac{1/2\pi}{\sqrt{C_0^2-c^2}} = \frac{1}{\pi\sqrt{C_0^2-c^2}}$$ para $-C_0 \le c \le C_0$

es decir

$$\boxed{f_{C_u}(c) = \frac{1}{\pi\sqrt{C_0^2 - c^2}}\,\Pi\left(\frac{c}{2C_0}\right)}$$

Propiedades: $\quad f_{C_u}(c) \geq 0 \quad$ y $\quad \int_{-\infty}^{\infty} f_{C_u}(c)\,dc = 1.$

para $|c| \leq C_0$, $f_{C_u}(c)$ siempre es positivo

para $|c| > C_0$ $\quad f_{C_u}(c) = 0$, cumple 1ª propiedad.

Con respecto a la segunda:

$$\int_{-\infty}^{\infty} f_{C_u}(c)\,dc = \int_{-C_0}^{C_0} \frac{1}{\pi\sqrt{C_0^2 - c^2}}\,dc = \frac{1}{\pi}\int_{-C_0}^{C_0} \frac{dc}{\sqrt{C_0^2 - c^2}}$$

Cambio de variable

$$u = \frac{c}{C_0} \qquad du = \frac{dc}{C_0}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{C_0\,du}{\sqrt{C_0^2 - C_0^2 u^2}} = \frac{1}{\pi}\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} =$$

$$= \frac{1}{\pi}\left[\arcsen u\right]_{-1}^{1} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

f) $\quad X_n = X(t_n) \quad$ con $\quad A(t) = A_0 \operatorname{sen}(2\pi f_a t)$

$$X_n = X(t_n) = A(t_n)\,C(t_n) = A_0 \operatorname{sen}(2\pi f_a t_n)\cdot C_0 \cos(2\pi f_0 t_n + \theta)$$

definiendo $\boxed{C_0' = A_0 C_0 \operatorname{sen}(2\pi f_a t_n)}$ se puede aplicar apartado anterior, recordando que $t_n$ es una constante al considerar variables aleatorias

Entonces $\boxed{f_{X_n}(x) = \frac{1}{\pi\sqrt{(C_0')^2 - x^2}} \, \Pi\left(\frac{x}{2C_0'}\right)} =$ (6)

$$\boxed{= \frac{1}{\pi\sqrt{A_0^2 C_0^2 \operatorname{sen}^2(2\pi f_a t_n) - x^2}} \, \Pi\left(\frac{x}{2A_0 C_0 \operatorname{sen}(2\pi f_a t_n)}\right)}$$

g) $E[X_n] = E[X(t_n)] = A_0 \operatorname{sen}(2\pi f_a t_n) \, E[C_n] = 0$, ya que como vimos en el apartado a) $E[C(t_n)] = E[C_n] = 0$.

Cómo la media no depende de $t_n$, es estacionaria con respecto a la media.

$$R_X(t_n, t_m) = E[X(t_n)X(t_m)] = E[A_0 \operatorname{sen}(2\pi f_a t_n) C(t_n) A_0 \operatorname{sen}(2\pi f_a t_m) C(t_m)]$$

$$= A_0^2 \operatorname{sen}(2\pi f_a t_n) \operatorname{sen}(2\pi f_a t_m) R_C(t_n, t_m)$$

por el apartado b)

$$R_C(t_n, t_m) = \frac{C_0^2}{2} \cos[2\pi f_0 (t_m - t_n)] = R_C(\tau) = \frac{C_0^2}{2} \cos(2\pi f_0 \tau)$$

$$R_X(t_n, t_m) = \left[\frac{A_0^2}{2} - \frac{A_0^2}{2} \cos[2\pi f_a (t_n + t_m)]\right] \frac{C_0^2}{2} \cos[2\pi f_0 (t_m - t_n)] =$$

$$= \frac{A_0^2 C_0^2}{4} \cos[2\pi f_0 (t_m - t_n)] - \frac{A_0^2 C_0^2}{4} \cos[2\pi f_a (t_n + t_m)] \cos[2\pi f_0 (t_m - t_n)]$$

Como tenemos dependencia con respecto a $t_n + t_m$ y no sólo con respecto a $\tau = t_m - t_n$ no es estacionaria con respecto a la autocorrelación. No es estacionaria en sentido amplio. Además por el apartado f) la función densidad de probabilidad depende de $t_n$. Tampoco en sentido estricto.