PROBLEMA 1.

(a) $X$ variable aleatoria Rayleigh con parámetro $\sigma$.

$$f_X(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} u(x)$$

$$\boxed{E[X]} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \quad \left| \begin{array}{l} t = \frac{x}{\sqrt{2}\sigma} \\ dt = \frac{dx}{\sqrt{2}\sigma} \end{array} \right|$$

$$= \int_0^{\infty} 2t^2 e^{-t^2} \sqrt{2}\sigma\, dt = 2\sqrt{2}\sigma \int_0^{\infty} t^2 e^{-t^2} dt \quad \left| \begin{array}{ll} u = t & du = dt \\ dv = te^{-t^2} & v = -\frac{e^{-t^2}}{2} \end{array} \right|$$

$$= 2\sqrt{2}\sigma \cancel{\left[ -\frac{t e^{-t^2}}{2} \right]_0^{\infty}} + \cancel{2}\sqrt{2}\sigma \int_0^{\infty} \frac{e^{-t^2}}{\cancel{2}} dt =$$

$$= \sqrt{2}\cdot\sigma \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{2}{2} \int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\sqrt{2\pi}\,\sigma \operatorname{erfc}(0) = \boxed{\sqrt{\frac{\pi}{2}}\sigma}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^{\infty} \frac{x^3}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \quad \left| \begin{array}{l} t = \frac{x}{\sqrt{2}\sigma} \\ dt = \frac{dx}{\sqrt{2}\sigma} \end{array} \right|$$

$$= \int_0^{\infty} \frac{2\sqrt{2}\sigma^3 t^3}{\sigma^2} e^{-t^2} \sqrt{2}\sigma\, dt = 4\sigma^2 \int_0^{\infty} t^3 e^{-t^2} dt \quad \left| \begin{array}{ll} u = t^2 & du = 2t\,dt \\ dv = te^{-t^2} dt & v = -\frac{e^{-t^2}}{2} \end{array} \right|$$

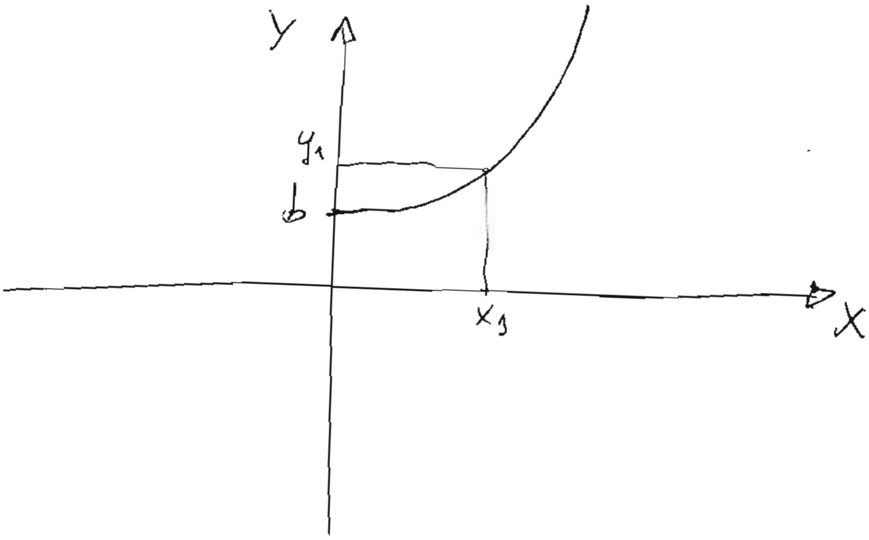
$$= 4\sigma^2 \cancel{\left[ -t^2 \frac{e^{-t^2}}{2} \right]_0^{\infty}} + 4\sigma^2 \int_0^{\infty} \frac{e^{-t^2}}{\cancel{2}} \cancel{2} t\, dt = 4\sigma^2 \int_0^{\infty} t e^{-t^2} dt = 4\sigma^2 \left[ -\frac{e^{-t^2}}{2} \right]_0^{\infty}$$

$$= 2\sigma^2$$

$$\boxed{\operatorname{Var}[X] = E[X^2] - E^2[X] = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = \sigma^2 \frac{4-\pi}{2}}$$

(b) $$\boxed{SNR(X)} = \frac{\left(\sqrt{\frac{\pi}{2}}\sigma\right)^2}{\sigma^2 \frac{4-\pi}{2}} = \frac{\frac{\pi}{\cancel{2}}\cancel{\sigma^2}}{\cancel{\sigma^2}\frac{4-\pi}{\cancel{2}}} = \boxed{\frac{\pi}{4-\pi}}$$

c) $Y = g(X) = b + cX^2$ 　　$X > 0$ por ser envolvente 　　②



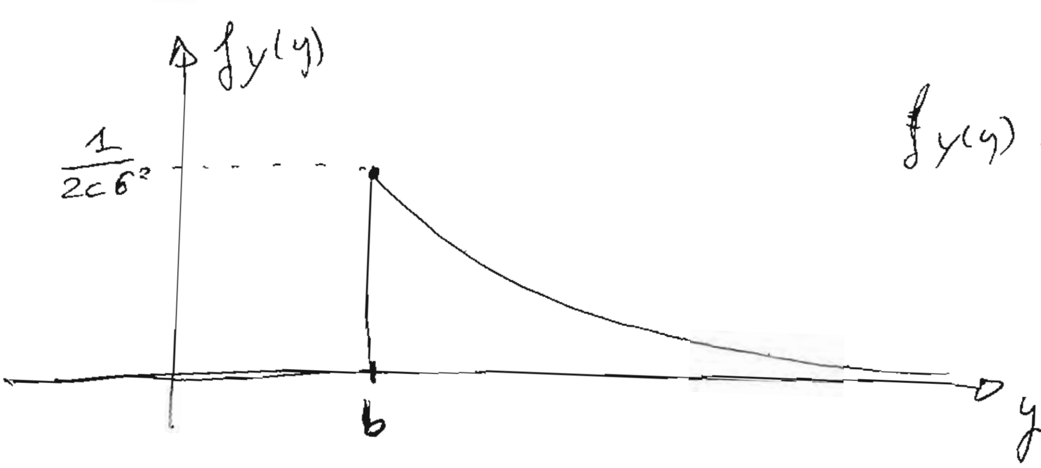

Una única solución.

$$y_1 = b + c x_1^2 \Rightarrow y_1 - b = c x_1^2 \Rightarrow x_1^2 = \frac{y_1 - b}{c} \qquad x_1 = +\sqrt{\frac{y_1 - b}{c}}$$

$$\left.\frac{dy}{dx}\right|_{x=x_1} = \left. 2cX \right|_{X=x_1} = 2c\sqrt{\frac{y_1-b}{c}} = 2\sqrt{c(y_1-b)}$$

$$\boxed{f_Y(y)} = \frac{f_X\left(\sqrt{\frac{y-b}{c}}\right)}{2\sqrt{c(y-b)}} = \frac{\sqrt{\frac{y-b}{c}}}{\sigma^2} \frac{\exp\left(-\frac{y-b}{2c\sigma^2}\right)}{2\sqrt{c(y-b)}} u(y-b)$$

$$= \boxed{\frac{1}{2c\sigma^2} \exp\left(-\frac{y-b}{2c\sigma^2}\right) u(y-b)}$$




$f_Y(y) \geq 0$

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_b^{\infty} \frac{1}{2c\sigma^2} \exp\left(-\frac{y-b}{2c\sigma^2}\right) dy = \frac{1}{2c\sigma^2}\int_b^{\infty} \exp\left(-\frac{y-b}{2c\sigma^2}\right) dy$$

$$= \frac{1}{2c\sigma^2}\left[-2c\sigma^2 \exp\left(-\frac{y-b}{2c\sigma^2}\right)\right]_b^{\infty} = \frac{2c\sigma^2}{2c\sigma^2} = 1.$$

(d) $$\boxed{\mathrm{PROB}(Y \geq 2b)} = \int_{2b}^{\infty} f_Y(y)\,dy = \int_{2b}^{\infty} \frac{1}{2c\sigma^2} \exp\left(-\frac{y-b}{2c\sigma^2}\right) dy$$

$$= \frac{1}{2c\sigma^2}\int_b^{\infty} \exp\left(-\frac{y-b}{2c\sigma^2}\right) dy = \frac{1}{\cancel{2c\sigma^2}}\left[-\cancel{2c\sigma^2} \exp\left(-\frac{y-b}{2c\sigma^2}\right)\right]_{2b}^{\infty} = \boxed{\exp\left(-\frac{b}{2c\sigma^2}\right)}$$

(e) $$\boxed{E[Y]} = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_b^{\infty} \frac{y}{2c\sigma^2} \exp\left(-\frac{y-b}{2c\sigma^2}\right) dy$$

$$= E\left[b + cX^2\right] = b + c\,E\left[X^2\right] = \boxed{b + 2c\sigma^2}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_b^{\infty} \frac{y^2}{2c\sigma^2} e^{-\frac{y-b}{2c\sigma^2}}\,dy \quad \left|\; t = \frac{y-b}{2c\sigma^2} \quad dt = \frac{dy}{2c\sigma^2} \right|$$

$$= \frac{1}{\cancel{2c\sigma^2}} \int_0^{\infty} (b + 2c\sigma^2 t)^2 e^{-t}\,dt \cdot \cancel{2c\sigma^2} = \int_0^{\infty} (b^2 + 4c\sigma^2 bt + 4c^2\sigma^4 t^2) e^{-t}\,dt$$

$$= b^2 \underbrace{\int_0^{\infty} e^{-t}\,dt}_{I_1} + 4c\sigma^2 b \underbrace{\int_0^{\infty} t e^{-t}\,dt}_{I_2} + 4c^2\sigma^4 \underbrace{\int_0^{\infty} t^2 e^{-t}\,dt}_{I_3}$$

$$I_1 = \int_0^{\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{\infty} = 1.$$

$$I_2 = \int_0^{\infty} t e^{-t}\,dt \quad \left|\; \begin{matrix} u = t & du = dt \\ dv = e^{-t}dt & v = -e^{-t} \end{matrix} \right| = \left[\cancel{te^{-t}}\right]_0^{\infty} + \overbrace{\int_0^{\infty} e^{-t}\,dt}^{I_1} = 1$$

$$I_3 = \int_0^{\infty} t^2 e^{-t}\,dt = \left|\; \begin{matrix} u = t^2 & du = 2t \\ dv = e^{-t}dt & v = -e^{-t} \end{matrix} \right| = \left[\cancel{t^2 e^{-t}}\right]_0^{\infty} + 2\underbrace{\int_0^{\infty} t e^{-t}\,dt}_{I_2} = 2$$

$$E[Y^2] = b^2 + 4c\sigma^2 b + 8c^2\sigma^4$$

(4)

$$\boxed{Var[Y]} = E[Y^2] - E^2[Y] = b^2 + 4c\sigma^2 b + 8c^2\sigma^4 - (b + 2c\sigma^2)^2$$

$$= \cancel{b^2} + \cancel{4c\sigma^2 b} + 8c^2\sigma^4 - \cancel{b^2} - 4c^2\sigma^4 - \cancel{4c\sigma^2 b} = \boxed{4c^2\sigma^4}$$

$$\boxed{SNR(Y) = \frac{(b+2c\sigma^2)^2}{4c^2\sigma^4} = \frac{b^2 + 4c^2\sigma^4 + 4c\sigma^2 b}{4c^2\sigma^4} = 1 + \frac{b^2}{4c^2\sigma^4} + \frac{b}{c\sigma^2}}$$

(f) $$\rho = \frac{Cov[X,Y]}{\sqrt{Var(X)\,Var(Y)}} \qquad Cov[X,Y] = E[XY] - E[X]E[Y]$$

$$E[X\cdot Y] = E[X(b+cX^2)] = E[bX + cX^3] = bE[X] + cE[X^3]$$

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \int_0^{\infty} \frac{x^4}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\,dx \left| \begin{array}{l} t = \frac{x}{\sqrt{2}\sigma} \\ dt = \frac{dx}{\sqrt{2}\sigma} \end{array} \right| = \int_0^{\infty} \frac{4\sigma^4 t^4}{\sigma^2} e^{-t^2} \sqrt{2}\sigma\,dt$$

$$= 4\sqrt{2}\sigma^3 \int_0^{\infty} t^4 e^{-t^2}\,dt \left| \begin{array}{ll} u = t^3 & du = 3t^2 \\ dv = te^{-t^2}dt & v = -\frac{e^{-t^2}}{2} \end{array} \right|$$

$$= 4\sqrt{2}\sigma^3 \cancel{\left[-t^3 \frac{e^{-t^2}}{2}\right]_0^{\infty}} + 4\sqrt{2}\sigma^3 \int_0^{\infty} \frac{3t^2 e^{-t^2}}{2}\,dt = 6\sqrt{2}\sigma^3 \int_0^{\infty} t^2 e^{-t^2}\,dt \left| \begin{array}{ll} u = t & du = dt \\ dv = te^{-t^2}dt & v = -\frac{e^{-t^2}}{2} \end{array} \right.$$

$$= 6\sqrt{2}\sigma^3 \cancel{\left[-t\frac{e^{-t^2}}{2}\right]_0^{\infty}} + 6\sqrt{2}\sigma^3 \int_0^{\infty} \frac{e^{-t^2}}{2}\,dt = 3\sqrt{2}\sigma^3 \frac{2}{2} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2}\,dt$$

$$= 3\frac{\sqrt{2\pi}}{2}\sigma^3 \,\mathrm{erfc}(0) = 3\sqrt{\frac{\pi}{2}}\sigma^3$$

$$E[XY] = b \cdot \sqrt{\frac{\pi}{2}}\sigma + c\,3\sqrt{\frac{\pi}{2}}\sigma^3 = \sqrt{\frac{\pi}{2}}\sigma\left[b + 3c\sigma^2\right]$$

$$Cov[X,Y] = E[XY] - E[X]E[Y] = \sqrt{\frac{\pi}{2}}\sigma\left[b + 3c\sigma^2\right] - \sqrt{\frac{\pi}{2}}\sigma \cdot (b + 2c\sigma^2)$$

$$= \sqrt{\frac{\pi}{2}}\sigma\left(\cancel{b} + 3c\sigma^2 - \cancel{b} - 2c\sigma^2\right) = \sqrt{\frac{\pi}{2}}c\sigma^3$$

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$$\boxed{\rho = \frac{Cov[X,Y]}{\sqrt{var(X)\,var(Y)}} = \frac{\sqrt{\frac{\pi}{2}}\,c\,\sigma^3}{\sqrt{\sigma^2\frac{4-\pi}{2}\,4c^2\sigma^4}} = \sqrt{\frac{\pi}{16-4\pi}}}$$

X e Y no están incorrelados.

(g) $z(t) = X\cos(2\pi f_0 t + \Theta) + Y\,sen(2\pi f_0 t + \Theta)$

$$E[z(t)] = E[X\cos(2\pi f_0 t + \Theta) + Y\,sen(2\pi f_0 t + \Theta)] =$$

$$= E[X]\,E[\cos(2\pi f_0 t + \Theta)] + E[Y]\,E[sen(2\pi f_0 t + \Theta)] = 0$$

$$R_Z(\tau) = E[z(t)\,z(t+\tau)] =$$

$$= E\Big[\big[X\cos(2\pi f_0 t + \Theta) + Y\,sen(2\pi f_0 t + \Theta)\big]\big[X\cos(2\pi f_0(t+\tau) + \Theta) + Y\,sen(2\pi f_0(t+\tau) + \Theta)\big]\Big]$$

$$= E[X^2]\,E[\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0(t+\tau) + \Theta)]$$

$$+ E[Y^2]\,E[sen(2\pi f_0 t + \Theta)\,sen(2\pi f_0(t+\tau) + \Theta)]$$

$$+ E[XY]\,E[sen(2\pi f t + \Theta)\cos(2\pi f_0(t+\tau) + \Theta) + \cos(2\pi f_0 t + \Theta)\,sen(2\pi f_0(t+\tau) + \Theta)]$$

$$= E[X^2]\cdot\frac{1}{2}\Big(E\big[\cos(2\pi f_0\tau) + \cancel{\cos(2\pi f(2t+\tau) + 2\Theta)}\big]\Big)$$

$$+ E[Y^2]\cdot\frac{1}{2}\Big(E\big[\cos(2\pi f_0\tau) - \cancel{\cos(2\pi f_0(2t+\tau) + 2\Theta)}\big]\Big)$$

$$+ E[XY]\,E[\cancel{sen(2\pi f_0(2t+\tau) + 2\Theta)}]$$

$$= \frac{1}{2}\cos(2\pi f_0\tau)\big[E[X^2] + E[Y^2]\big] = \boxed{\frac{2\sigma^2 + b^2 + 4c\sigma^2 b + 8c^2\sigma^4}{2}\cos(2\pi f_0\tau)}$$

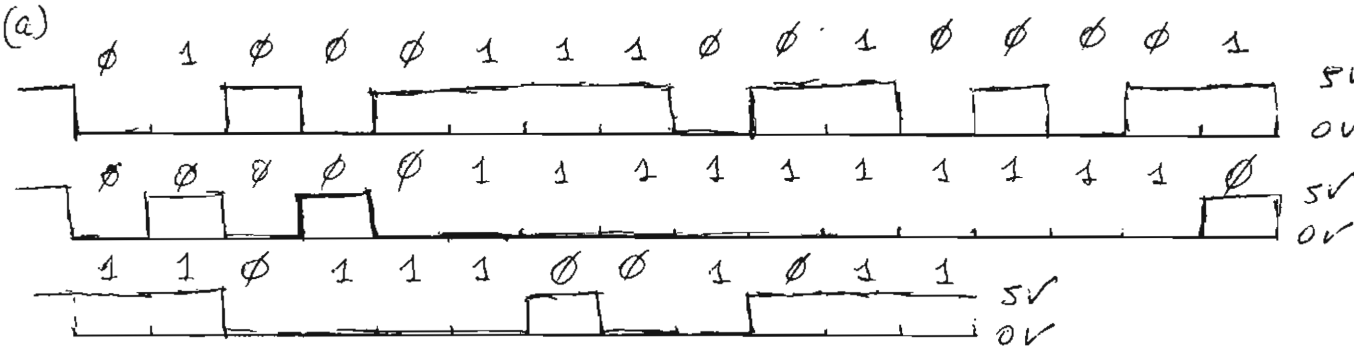
Sí es estacionaria.

$$\boxed{S_Z(f) = \frac{2\sigma^2 + b^2 + 4c\sigma^2 b + 8c^2\sigma^4}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right]}$$

PROBLEMA 2.

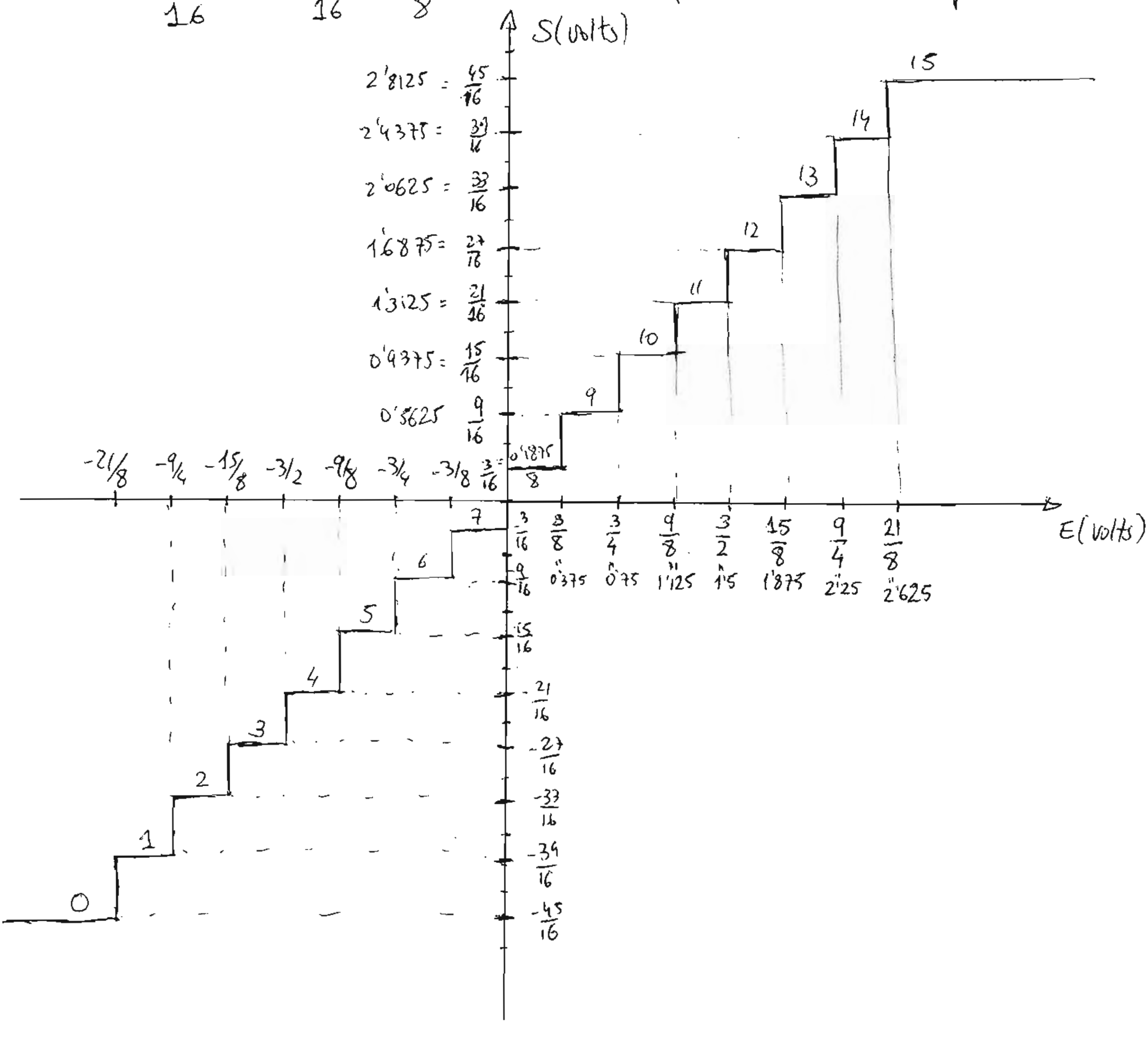
0100011100100001 | 0000011111111110 | 110111001011

(a)



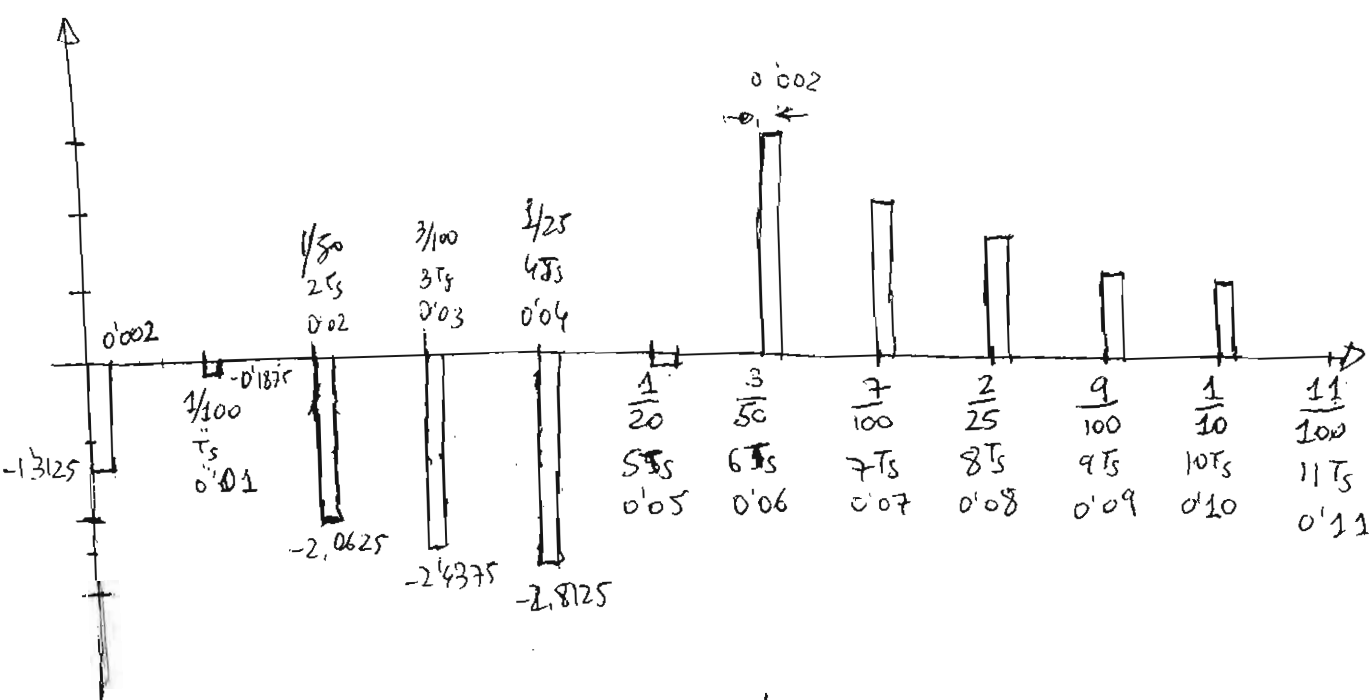

(b) $n = 4$ bits $\Rightarrow L = 2^n = 2^4 = 16$ niveles $\quad \Delta_{max} = 3$

$$\delta = \frac{2 \cdot \Delta_{max}}{16} = \frac{2 \cdot 3}{16} = \frac{3}{8} = 0'375$$ tipo mid-threas (L par)



(c) Agrupando los bits de 4 en 4:

| | | | |
|---|---|---|---|
| 0100 | 4 | -21/16 | -1,3125 |
| 0111 | 7 | -3/16 | -0,1875 |
| 0010 | 2 | -33/16 | -2,0625 |
| 0001 | 1 | -39/16 | -2,4375 |
| 0000 | 0 | -45/16 | -2,8125 |
| 0111 | 7 | -3/16 | -0,1875 |
| 1111 | 15 | 45/16 | 2,8125 |
| 1110 | 14 | 39/16 | 2,4375 |
| 1101 | 13 | 33/16 | 2,0625 |
| 1100 | 12 | 27/16 | 1,6875 |
| 1011 | 11 | 21/16 | 1,3125 |




(d) Normalizar valores con respecto a Amax

| | | |
|---|---|---|
| 4 | -1'3125 | -0'4375 |
| 7 | -0'1875 | -0'0625 |
| 2 | -2,0625 | -0'6875 |
| 1 | -2,4375 | -0'8125 |
| 0 | -2,8125 | -0'9375 |
| 7 | -0,1875 | -0'0625 |
| 15 | 2,8125 | 0'9375 |
| 14 | 2,4375 | 0'8125 |
| 13 | 2,0625 | 0'6875 |
| 12 | 1,6875 | 0'5625 |
| 11 | 1,3125 | 0'4375 |

log A:

$$|V_2| = \begin{cases} \frac{A|V_1|}{1+\ln(A)} & 0 \leq |V_1| \leq \frac{1}{A} \quad (1) \\ \frac{1+\ln(A|V_1|)}{1+\ln(A)} & \frac{1}{A} \leq |V_1| \leq 1 \quad (2) \end{cases}$$

Para (1) $|V_1| = \frac{1+\ln(A)}{A}|V_2| = 0'0466|V_2|$

Para (2)

$$\ln(A|V_1|) = |V_2|\,(1+\ln(A)) - 1$$

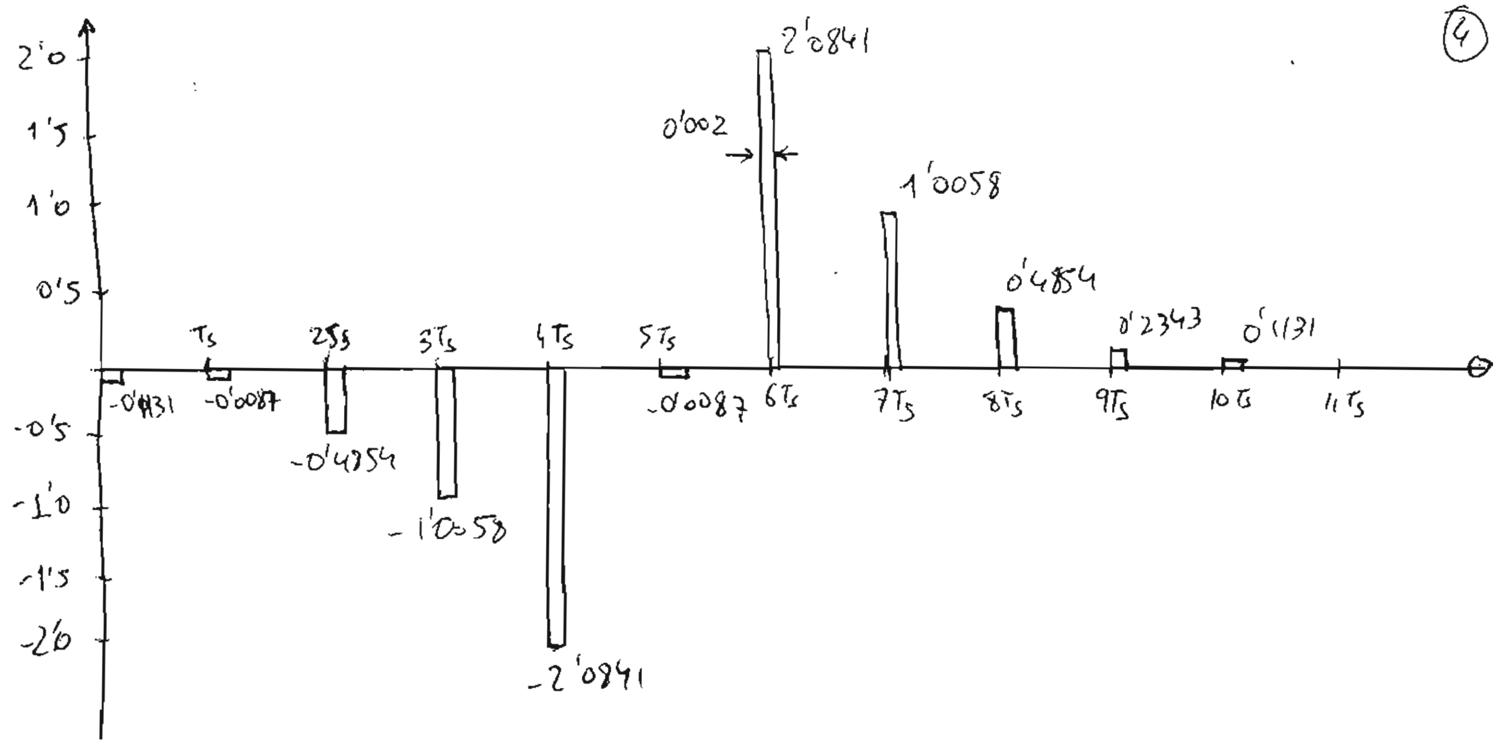
$$A|V_1| = \exp((1+\ln(A))|V_2|) \cdot e^{-1} \Rightarrow |V_1| = \frac{\exp((1+\ln(A))|V_2|)}{e \cdot A}$$

$$|V_1| = \frac{\exp(5'8283|V_2|)}{339'78}$$

Entonces:

$$|V_1| = \begin{cases} \frac{\exp(5'8283|V_2|)}{339'78} & \text{si } \frac{\exp(5'8283|V_2|)}{339'78} > \frac{1}{A} = 8 \cdot 10^{-3} \\ 0'0466|V_2| & \text{en otro caso} \end{cases}$$

| | | | |
|---|---|---|---|
| 4 | $-0'0377$ (2) log | Multiplicando por $A_{max}$ | $-0'1131$ |
| 7 | $-0'0029$ (1) lin | | $-0'0087$ |
| 2 | $-0'1618$ (2) log | | $-0'4854$ |
| 1 | $-0'3353$ (2) log | | $-1'0058$ |
| 0 | $-0'6947$ (2) log | | $-2'0841$ |
| 7 | $-0'0029$ (1) lin | | $-0'0087$ |
| 15 | $0'6947$ (2) log | | $2'0841$ |
| 14 | $0'3353$ (2) log | | $1'0058$ |
| 13 | $0'1618$ (2) log | | $0'4854$ |
| 12 | $0'0781$ (2) log | | $0'2343$ |
| 11 | $0'0377$ (2) log | | $0'1131$ |




(e) Para el escalón más pequeño se usa la ~~zona~~ lineal

$$|V_2| = \frac{A|V_1|}{1+\ln(A)}$$

el tamaño del escalón es proporcional al inverso de la derivada

$$\frac{d|V_2|}{d|V_1|} = \frac{A}{1+\ln(A)} \quad \text{cte} \qquad \text{tamaño escalón} \quad \frac{1+\ln(A)}{A}$$

Ganancia = inverso tamaño escalón: $\frac{A}{1+\ln(A)} = \frac{125}{1+\ln(125)} = 21'44$

en dB: $\boxed{20\log_{10} 21'44 = 26'6273 \text{ dB}}$

para señales potentes, estamos en la zona logarítmica:

$$\frac{d|V_2|}{d|V_1|} = \frac{1}{(1+\ln(A))|V_1|} \qquad \text{tamaño escalón} \quad (1+\ln(A))|V_1|$$

tamaño máximo, para $|V_1| = 1$ $\quad 1+\ln(A)$

Pérdida: inverso tamaño escalón: $\frac{1}{1+\ln(A)} = \frac{1}{1+\ln(125)} = 0'1716$

en dB $\quad 20\log_{10} 0'1716 = \boxed{-15'3109 \text{ dB}}$

(5)

(f) NRZ unipolar

$S_W(f) = \frac{N_0}{2}$.

$B = 5000$ Hz

$A_{rx} = 5 \cdot 10^{-5}$ V

$N_0 = 3 \cdot 10^{-15}$ W/Hz

$$\sigma_N^2 = \frac{N_0}{2} \cdot 2B = N_0 \cdot B = 3 \cdot 10^{-15} \cdot 5 \cdot 10^{3} = 1'5 \cdot 10^{-11} \text{ W}.$$

[Caso visto en teoría]

$$P_e = \frac{1}{2} \operatorname{erfc}\left( \frac{5 \cdot 10^{-5}}{2\sqrt{2} \cdot \sqrt{1'5 \cdot 10^{-11}}} \right) = \frac{1}{2} \operatorname{erfc}(4'5644)$$

$u = 4'5644 > 1'5$ se puede usar la aproximación:

$$\boxed{P_e \simeq \frac{1}{2} \frac{\exp(-4'5644^2)}{4'5644\sqrt{\pi}} = 5'53 \cdot 10^{-11}}$$