PROBLEMA 1

1. $E[Z] = E[X_1^2 + X_2^2] = E[X_1^2] + E[X_2^2]$

sabemos que $E[X_1] = 0$, $E[X_2] = 0$, $\mathrm{var}[X_1] = \sigma_1^2$, $\mathrm{var}[X_2] = \sigma_2^2$

y que
$$E[X_1^2] = \mathrm{var}[X_1] + E^2[X_1] = \sigma_1^2$$
$$E[X_2^2] = \mathrm{var}[X_2] + E^2[X_2] = \sigma_2^2$$

entonces $\boxed{E[Z] = \sigma_1^2 + \sigma_2^2}$

$\mathrm{Var}[Z] = E[Z^2] - E^2[Z]$ nos falta determinar $E[Z^2]$

$$E[Z^2] = E[(X_1^2 + X_2^2)^2] = E[X_1^4 + X_2^4 + 2X_1^2X_2^2] = E[X_1^4] + E[X_2^4] + 2E[X_1^2X_2^2]$$

para una variable Gaussiana X con media cero y varianza σ^2,

$$E[X^{2n}] = \frac{(2n)!}{2^n\, n!}\sigma^{2n}$$

para $n=2$: $$E[X^4] = \frac{4!}{2^2\, 2!}\sigma^4 = \frac{4\cdot 3\cdot 2}{4\cdot 2}\sigma^4 = 3\sigma^4$$

entonces para nuestro caso:
$$E[X_1^4] = 3\sigma_1^4 \quad y \quad E[X_2^4] = 3\sigma_2^4$$

Además por ser X_1 y X_2 independientes se tiene que, X_1^2 y X_2^2 son también ind.

$$E[X_1^2 X_2^2] = E[X_1^2]\cdot E[X_2^2] = \sigma_1^2\sigma_2^2$$

En definitiva:
$$E[Z^2] = 3\sigma_1^4 + 3\sigma_2^4 + 2\sigma_1^2\sigma_2^2$$

y
$$\boxed{\mathrm{var}[Z^2]} = E[Z^2] - E^2[Z] = 3\sigma_1^4 + 3\sigma_2^4 + 2\sigma_1^2\sigma_2^2 - (\sigma_1^2 + \sigma_2^2)^2 =$$
$$3\sigma_1^4 + 3\sigma_2^4 + \cancel{2\sigma_1^2\sigma_2^2} - \sigma_1^4 - \sigma_2^4 - \cancel{2\sigma_1^2\sigma_2^2} = \boxed{2(\sigma_1^4 + \sigma_2^4)}$$

② $\rho_{Z,X_1} = \dfrac{cov(Z,X_1)}{\sqrt{var(Z)\cdot var(X_1)}}$ donde $cov(Z,X_1) = E[Z\cdot X_1] - E[Z]\cdot E[X_1]$

pero como $E[X_1]=0$ entonces:

$$cov(Z,X_1) = corr(Z,X_1) = E[Z\cdot X_1]$$

desarrollando:

$$E[Z\cdot X_1] = E[(X_1^2+X_2^2)X_1] = E[X_1^3 + X_1\cdot X_2^2] = E[X_1^3] + E[X_1\cdot X_2^2]$$

pero para una variable gaussiana X_1 con media cero y varianza σ^2, los momentos impares son cero, por lo que $E[X_1^3]=0$.

Además por ser X_1 y X_2 independientes:

$$E[X_1\cdot X_2^2] = E[X_1]\cdot E[X_2^2] = 0\cdot E(X_2^2) = 0 \quad \text{pues } E[X_1]=0.$$

En resumen:

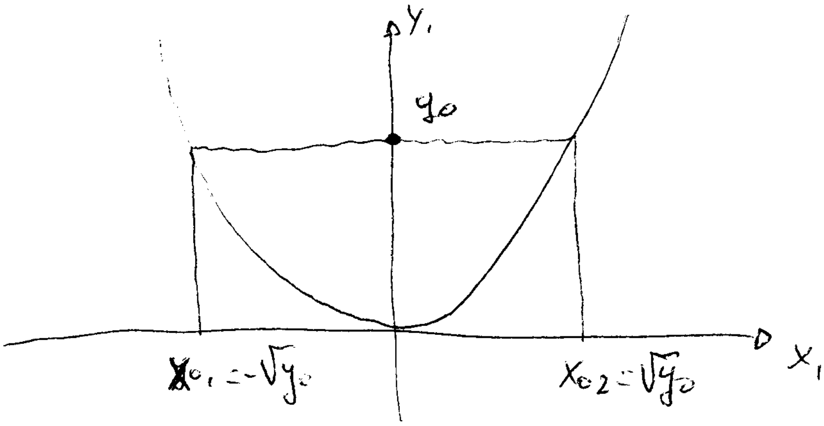
$$cov[Z,X_1] = E[ZX_1] = 0 \Rightarrow \boxed{\rho_{Z,X_1}=0}$$ y por ser Gaussianas Z y X_1 serán además variables independientes.

Igualmente se cumple para Z y X_2: $\boxed{\rho_{Z,X_2}=0}$ y son independientes

③ Vamos a determinar en primer lugar $f_{Y_1}(y_1)$ y $f_{Y_2}(y_2)$ para $Y_1=X_1^2$ e $Y_2=X_2^2$. Por simetría $f_{Y_1}(y)=f_{Y_2}(y)$ serán iguales puesto que X_1 e X_2 tienen la misma distribución y nos dicen que $\sigma_1^2=\sigma_2^2=\sigma^2$.

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right) = f_{X_2}(x)$$

La transformación $Y_1 = X_1^2$ tiene la gráfica



para un y_0 dado tenemos dos raíces $x_{01} = -\sqrt{y_0}$ y $x_{02} = \sqrt{y_0}$, además los y sólo están definidos para valores positivos por lo que $f_{Y_1}(y_1) = 0$ para $y_1 < 0$. Aplicando el teorema fundamental de la probabilidad:

$$f_{Y_1}(y) = \frac{f_{X_1}(x_{01})}{\left|\frac{dy_1}{dx_1}\right|_{x_1 = x_{01}}} + \frac{f_{X_1}(x_{02})}{\left|\frac{dy_1}{dx_1}\right|_{x_1 = x_{02}}}$$

$$\frac{dy_1}{dx_1} = 2x_1 \qquad \left|\frac{dy_1}{dx_1}\right| = 2|x_1| \qquad \left|\frac{dy_1}{dx_1}\right|_{x = x_{01}} = 2\sqrt{y} = \left|\frac{dy_1}{dx_1}\right|_{x = x_{02}}$$

$$\boxed{f_{Y_1}(y)} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\sqrt{y})^2}{2\sigma^2}\right) \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\sqrt{y})^2}{2\sigma^2}\right) \frac{1}{2\sqrt{y}}$$

$$\boxed{= \frac{1}{\sqrt{2\pi y}\,\sigma} \exp\left(-\frac{y}{2\sigma^2}\right) u(y)}$$

Por simetría:

$$f_{Y_2}(y) = \frac{1}{\sqrt{2\pi y}\,\sigma} \exp\left(-\frac{y}{2\sigma^2}\right) u(y)$$

Finalmente sumamos Y_1 e Y_2 : $Z = Y_1 + Y_2$ se sabe que

$$\boxed{f_Z(z) = f_{Y_1}(z) * f_{Y_2}(z) = \int_{-\infty}^{\infty} f_{Y_1}(y)\, f_{Y_2}(z-y)\, dy}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi y}\,\sigma} \exp\left(-\frac{y}{2\sigma^2}\right) u(y) \frac{1}{\sqrt{2\pi (z-y)}\,\sigma} \exp\left(-\frac{z-y}{2\sigma^2}\right) u(z-y)\, dy$$

$$= \frac{1}{2\pi\sigma^2} \int_0^z \frac{1}{\sqrt{y(z-y)}} \exp\left(-\frac{\cancel{y}+z-\cancel{y}}{2\sigma^2}\right) dy \qquad \underline{z>0}$$

$$= \frac{\exp\left(\frac{-z}{2\sigma^2}\right)}{2\pi\sigma^2} \int_0^z \frac{1}{\sqrt{yz-y^2}}\, dy$$

tengo que formar un cuadrado en el denominador:

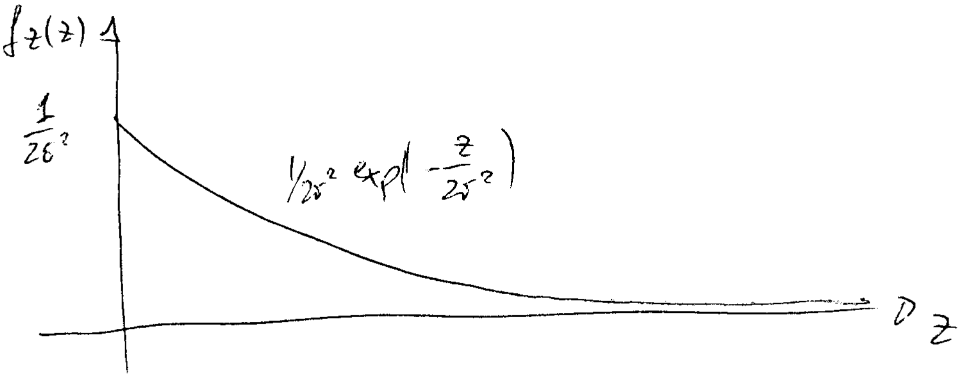
$$\left(y-\frac{z}{2}\right)^2 = y^2 - yz + \frac{z^2}{4} \quad \text{entonces} \quad yz - y^2 = \frac{z^2}{4} - \left(y-\frac{z}{2}\right)^2$$

$$= \frac{\exp\left(-\frac{z}{2\sigma^2}\right)}{2\pi\sigma^2} \int_0^z \frac{1}{\sqrt{\frac{z^2}{4} - \left(y-\frac{z}{2}\right)^2}}\, dy \quad \left| \begin{array}{l} u = y - \frac{z}{2} \\ du = dy \end{array} \right| \begin{array}{ll} u=0 & y=-\frac{z}{2} \\ u=z & y=\frac{z}{2} \end{array}$$

$$= \frac{\exp\left(-\frac{z}{2\sigma^2}\right)}{2\pi\sigma^2} \int_{-z/2}^{z/2} \frac{1}{\sqrt{\frac{z^2}{4} - u^2}}\, du = \frac{\exp\left(-\frac{z}{2\sigma^2}\right)}{2\pi\sigma^2} \left[\operatorname{asin}\left[\frac{2u}{z}\right]\right]_{-z/2}^{z/2}$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) \left[\operatorname{asin}(1) - \operatorname{asin}(-1)\right) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{\cancel{\pi}}{2\cancel{\pi}\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) = \boxed{\frac{1}{2\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) u(z)}$$



Comprobación área unidad:

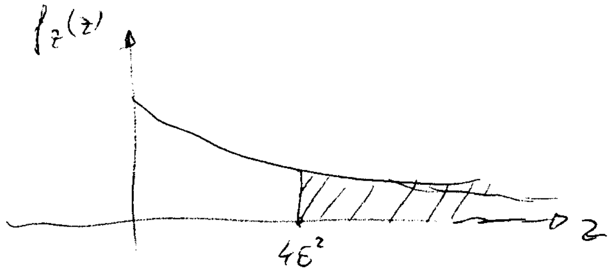
$$\int_{-\infty}^{\infty} f_Z(z)\,dz = \int_0^{\infty} \frac{1}{2\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) dz = \frac{1}{2\sigma^2}\left[\frac{\exp\left(-\frac{z}{2\sigma^2}\right)}{-\frac{1}{2\sigma^2}}\right]_0^{\infty}$$

$$= \left[-\exp\left(-\frac{z}{2\sigma^2}\right)\right]_0^{\infty} = \left[0-(-1)\right] = 1.$$

(4) $\text{PROB}\left(Z > E[Z] + \sqrt{Var[Z]}\right)$ $\qquad \sigma_1^2 = \sigma_2^2 = \sigma^2$

$$E[Z] = \sigma^2 + \sigma^2 = 2\sigma^2$$

$$Var[Z] = 2(\sigma^4 + \sigma^4) = 4\sigma^4$$



$$E[Z] + \sqrt{Var[Z]} = 2\sigma^2 + 2\sigma^2 = 4\sigma^2$$

$$\int_{4\sigma^2}^{\infty} \frac{1}{2\sigma^2} \exp\left(-\frac{z}{2\sigma^2}\right) dz = \frac{1}{2\sigma^2}\left[\frac{\exp\left(-z/2\sigma^2\right)}{-1/2\sigma^2}\right]_{4\sigma^2}^{\infty} = \left[0 - \left(-\exp\left(\frac{-4\sigma^2}{2\sigma^2}\right)\right)\right]$$

$$= \exp(-2) = e^{-2} = \boxed{0{,}14}$$

(5) $\boxed{E[X(t)]} = E\left[X_1 \cos(2\pi f_1 t) + X_2 \cos(2\pi f_2 t) + Z \cos(2\pi(f_1+f_2)t)\right]$

$$= \cancel{E[X_1]} \cos(2\pi f_1 t) + \cancel{E[X_2]} \cos(2\pi f_2 t) + E[Z] \cos(2\pi(f_1+f_2)t)$$

$$= \boxed{(\sigma_1^2 + \sigma_2^2) \cos(2\pi(f_1+f_2)t)}$$

$$R_X(t_1,t_2) = E\Big[\big(X_1\cos(2\pi f_1 t_1) + X_2\cos(2\pi f_2 t_1) + Z\cos(2\pi(f_1+f_2)t_1)\big)\cdot$$

$$\cdot\big(X_1\cos(2\pi f_1 t_2) + X_2\cos(2\pi f_2 t_2) + Z\cos(2\pi(f_1+f_2)t_2)\big)\Big]$$

$$= E[X_1^2]\cos(2\pi f_1 t_1)\cos(2\pi f_1 t_2) + E[X_2^2]\cos(2\pi f_2 t_1)\cos(2\pi f_2 t_2)$$

$$+ E[Z^2]\cos(2\pi(f_1+f_2)t_1)\cos(2\pi(f_1+f_2)t_2)$$

puesto que $E[X_1X_2] = E(X_1)\cdot E[X_2] = 0$

$$E[X_1Z] = E[X_2Z] = 0$$

$$\boxed{\begin{aligned} R_X(t_1,t_2) = {} & \sigma_1^2\left[\frac{1}{2}\cos(2\pi f_1(t_1-t_2)) + \frac{1}{2}\cos(2\pi f_1(t_1+t_2))\right] \\ & + \sigma_2^2\left[\frac{1}{2}\cos(2\pi f_1(t_1-t_2)) + \frac{1}{2}\cos(2\pi f_1(t_1+t_2))\right] \\ & + (3\sigma_1^4 + 3\sigma_2^4 + 2\sigma_1^2\sigma_2^2)\left[\frac{1}{2}\cos(2\pi(f_1+f_2)(t_1-t_2)) + \frac{1}{2}\cos(2\pi(f_1+f_2)(t_1+t_2))\right] \end{aligned}}$$

No es estacionario puesto que la media no es constante y depende de t y la autocorrelación no sólo depende de t_1-t_2, sino también de t_1+t_2.

PROBLEMA 2.

1) AM ~~85%~~ $f_c = 800\,kHz$ $\max\{|m(t)|\} = 5$ $P_M^{max} = 24\,W$

$$S_{max}^{AM}(t) = A_c[1 + K_a \max\{|m(t)|\}]\cos(2\pi f_c t)$$

$$K_a \cdot \max\{|m(t)|\} = 0'85 = 5 \cdot K_a \quad \boxed{K_a = \frac{0'85}{5} = 0'17}$$

$$S_{max}^{AM}(t) = A_c[1 + 0'85]\cos(2\pi f_c t) = 1'85 \cdot A_c \cos(2\pi f_c t)$$

$$P_M^{max} = 1'85^2 \cdot A_c^2 \cdot 1/2 \Rightarrow A_c^2 = \frac{24 \cdot 2}{1'85^2} = 14,02 \; (A_c = 3'745)$$

La potencia media de la moduladora:

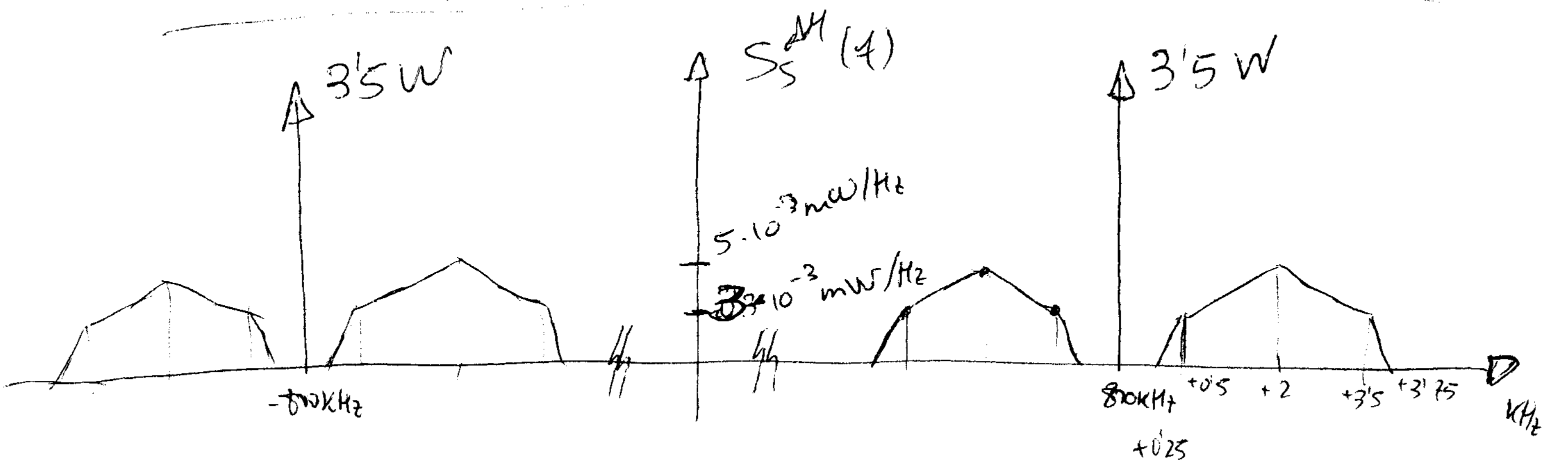
$$\boxed{P_M = \int_{-\infty}^{\infty} S_M(f)\,df} = 4 \cdot \left(\frac{250 \cdot 0,03}{2}\right) + 2 \cdot \left(3000 \cdot 0,03\right) + 2\left(\frac{3000 \cdot 0,02}{2}\right)$$

$$= 15 + 180 + 60 = 255\,mW = \boxed{0'255\,W}$$

a) $$\boxed{P_S^{AM} = \frac{A_c^2}{2}(1 + K_a^2 P_M) = \frac{14,02}{2}(1 + 0'17^2 \cdot 0'255) = 7,064\,W}$$

$$\boxed{P_C = \frac{A_c^2}{2} = 7,01\,W} \qquad \boxed{P_{LSB} = P_{USB} = \frac{A_c^2}{4} K_a^2 P_M = 0,026\,W}$$

b) $$S_S^{AM}(f) = \frac{A_c^2}{4}\delta(f - f_c) + \frac{A_c^2}{4}\delta(f + f_c) + \frac{A_c^2 K_a^2}{4} S_M(f - f_c) + \frac{A_c^2 K_a^2}{4} S_M(f + f_c)$$



$\boxed{DSB}$ $S_{max}^{DSB}(t) = A_c \cos(2\pi f_c t) \cdot max(f_m(t)) = 5 A_c \cos(2\pi f_c t)$

a)

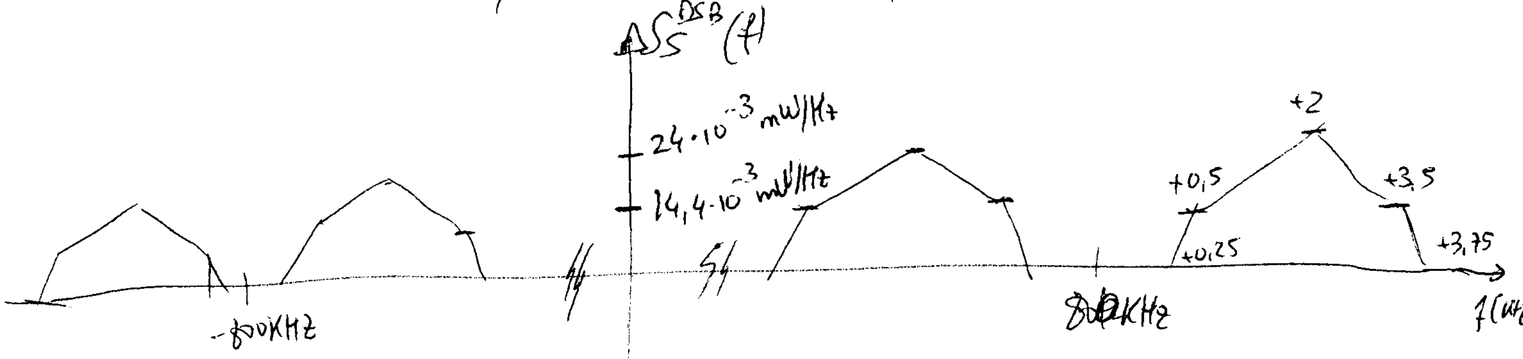
$$24 = 25 \cdot A_c^2 \frac{1}{2} \qquad A_c^2 = \frac{24 \cdot 2}{25} = 1'92 \quad (A_c = 1,386)$$

$$\boxed{P_S = \frac{A_c^2}{2} P_M = \frac{1'92}{2} 0,255 = 0,2448\ W}$$

$P_c = 0$

$$\boxed{P_{USB} = P_{LSB} = 0,1224\ W}$$

b) $S_S^{DSB}(f) = \frac{A_c^2}{4} S_M(f - f_c) + \frac{A_c^2}{4} S_M(f + f_c)$



2. $\boxed{P_T = P_c = 24\ W = P_{máxima}} = \frac{A_c^2}{2}$ $\qquad W = 3750\ Hz$

ANCHO DE BANDA. $\qquad D = 6$

CARSON. $\boxed{B_W^{CARSON} = 2W(1 + D) = 2 \cdot 3750 (1 + 6) = 52500\ Hz}$

1% $\boxed{B_W^{1\%} = 2 \cdot n_{max} \cdot W = 2 \cdot 9 \cdot 3750 = 67500\ Hz}$

↳ mirando en la tabla $n_{max} = 9$ ($D = 6$)

$$\boxed{B_W = \frac{B_W^{CARSON} + B_W^{1\%}}{2} = 60\ KHz}$$

(2)

3. Suponiendo que la señal se ha atenuado un factor K.

$P_R = \frac{A_c^2}{2} \cdot \frac{1}{K}$ (en las fórmulas donde ponía A_c^2 sustituyo por $\frac{A_c^2}{K}$)

$$SNR_o^{FM} = \frac{3 \cdot A_c^2 \cdot K_f^2 \cdot P_M}{2 \cdot W^3 \cdot N_0 \cdot K}$$

$$SNR_o^{AM} = \frac{A_c^2 \cdot K_a^2 P_M}{2W \cdot N_0 \cdot K}$$

$$SNR_o^{DSB} = \frac{A_c^2 \cdot P_M}{2W \cdot N_0 \cdot K}$$

A_c^2 lo conozco en cada caso, P_M también, W también y en AM conozco K_a. No sé cuánto vale $N_0 \cdot K$, la determino a partir de SNR_o^{FM}. Me falta determinar K_f.

$\Delta f = K_f \max\{m(t)\}$ $\quad D = \frac{\Delta f}{W}$ $\quad \boxed{\Delta f = D \cdot W = 6 \cdot 3750 = 22500 \, Hz}$

$$K_f = \frac{\Delta f}{\max\{|m(t)|\}} = \frac{22500}{5} = 4500 \; V/Hz$$

$$SNR_o^{FM} = 60 dB \equiv 10^{60/10} = 10^6$$

$$N_0 \cdot K = \frac{3 A_c^2 K_f^2 P_M}{2 \cdot W^3 \cdot SNR_o^{FM}} = \frac{3 \cdot 48 \cdot 4500^2 \cdot 0'255}{2 \cdot 3750^3 \cdot 10^6} = 7{,}05 \cdot 10^{-9} \; W/Hz.$$

DSB

$$\boxed{SNR_o^{DSB} = \frac{1'92 \cdot 0'255}{2 \cdot 3750 \cdot 7'05 \cdot 10^{-9}} = 9259{,}26 \equiv 39{,}666 \, dB}$$

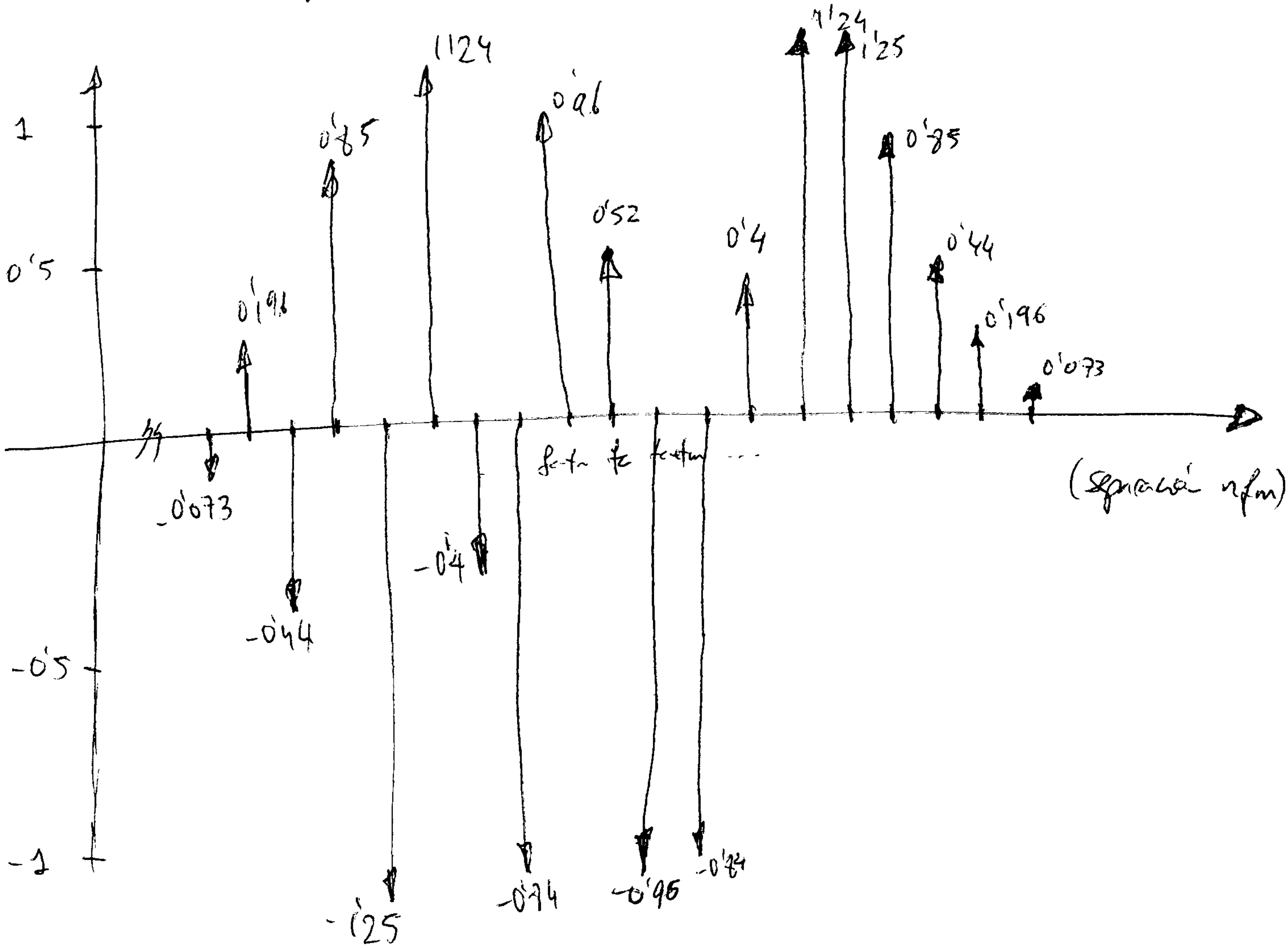
AM

$$\boxed{SNR_o^{AM} = \frac{14{,}02 \cdot 0'17^2 \cdot 0{,}255}{2 \cdot 3750 \cdot 7'05 \cdot 10^{-9}} = 1954{,}66 \equiv 32{,}91 \, dB}$$

4. $f_m = 5KHz$ $D=6=\beta$ $Ac=\sqrt{48}=6'93$

Dibujamos sólo frecuencias positivas (el módulo es par la fase es impar)

$$S_{FM}(f) = \frac{Ac}{2}\sum_{n=-\infty}^{\infty} J_n(D)\left[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\right]$$



Hemos usado la propiedad de $J_n(\beta) = -J_{-n}(\beta)$